DEPARTMENT COMMERCE

THE HYPERGEOMETRIC AND LEGENDRE FUNCTIONS WITH APPLICATIONS TO INTEGRAL EQUATIONS OF POTENTIAL

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STANDARDS

Preface

This is an outline of the theory of, and a collection of formulas pertaining to, the ordinary hypergeometric function with especial reference to the associated Legendre functions. The linear and quadratic transformations and analytic continuations of the hypergeometric function of z to all parts of the z-plane are written out at great length and for unrestricted values of its three parameters. Section VI is an attempt to make accessible those transformations and properties of associated Legendre of general argument and parameters which are most commonly required in physics, without trespassing upon the proper ground of treatises devoted to the theory of these functions.

Perhaps the most natural formal extension is the "generalized" hypergeometric function of z with more than three parameters this being defined as the solution of a differential equation having three singular points as before but of higher order than the second. However if it is felt

that, for some reason, the study of solutions of second-order differential equations is more urgent then the next in line is Heun's function of z with six parameters which presents itself as a solution of a Fuchsian equation of second order having four singular points. This function is the continuation of a power series in z whose coefficients cannot be written out explicitly, being themselves solutions of a difference-equation of second order. This generalization is the more difficult of the two but methods are available as in the ordinary case for obtaining all desired analytic continuations. Instead of six there are now twenty-four homographic transformations of the independent variable which interchange three of the singular points among the four so that a richer variety of relationships is obtained. Some of these analytic continuations are given in section VII. They are utilized in the last application of section X in the construction of certain normal functions which are solutions of the Lamé-Wangerin differential equation. By suitable choice of the

Bernoulli parameter, Heun's Series becomes a finite polynomial which is a solution of this equation; the Lamé-Hermite polynomials are similarly obtained as special cases of this function.

The theory necessary for the applications in section X is given in VIII and IX. There are three features of this which serve to unify the examples. The first is the analogy between the two-dimensional, logarithmic potential of simple distributions and the two-dimensional "potential" which is here called reduced potential. This potential arises when the boundary values are given on surfaces of revolution although these values and the resulting potential are not restricted to the case of axial symmetry.

The second idea, scarcely separable from the first, is that of the role of the Legendre function $Q_{m-\frac{1}{2}}\left(1+\frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1}\right)$, analogous to $-\log\sqrt{(x-x_1)^2+(y-y_1)^2}$. It appears as the symmetric nucleus of integral equations of potential theory. Its canonical expansions in normal functions, or its integral representations, in various systems of separable coordinates amount in each case to a new "addition-theorem" and furnish the key to

the formal solution of a certain class of problems.
The third idea is that of the infinitely many spatial interpretations of any reduced potential which follows from the invariance of its partial differential equation to a real homographic transformation, this being essentially an inversion in a circle centered on the axis of symmetry.

In section X two simple classes of boundary-value problems are illustrated in which (α, β) are "separable" coordinates for the potential equation. In one the potential is prescribed, say $f(\alpha)$, upon a surface of revolution whose generator is a member of the family of meridian curves, $\beta =$ constant (toroids, spheres, ellipsoids). The solution depends upon the development of $f(\alpha)$ in a series of normal functions of α which are solutions of an ordinary differential equation of second order with α as independent variable. These elementary cases including also the annular coordinates furnish examples of the Sturm-Liouville theory. In the other boundary-value problem that theory may be inapplicable, for if the potential, say $f(\beta)$, is assigned on a surface whose trace

is a member of the orthogonal family, α = constant, an integral representation of $f(\beta)$ analogous to Fourier's integral may be required in which the development-function satisfies a second-order differential equation with β as independent variable. The problem thus presented is considered in section VIII where a representation of an arbitrary function is obtained in the form of a double integral in which the development-functions satisfy a second-order differential equation of somewhat general form. The simplest special case is reducible to Mellin's form of Fourier's integral, and the particular cases applied here are integral representations of a given function in terms cylinder functions and of a variety of associated Legendre functions T_ν^μ, P_ν^μ and Q_ν^μ where the (complex) integration may be with respect to either an upper or a lower parameter.

The Hypergeometric and Legendre functions with applications to Integral Equations of potential theory.

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Contents.

Transformations of the Hypergeometric Function.

I Definitions and preliminary formulas

The argument of the hg function is the complex variable $z \equiv x+iy$, its three parameters α, β, γ, being also complex in general.

If γ is not a negative integer or zero and if $|z|<1$ the hg function is defined by the hg series

$$F(\alpha,\beta,\gamma;z) = 1 + \frac{\alpha.\beta}{\gamma}\frac{z}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \cdots\cdots$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{s=0}^{\infty} z^s \frac{\Gamma(s+\alpha)\Gamma(s+\beta)}{\Gamma(s+1)\Gamma(s+\gamma)}$$

$$= \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\beta)}\sum_{0}^{\infty}(-1)^s z^s \frac{\Gamma(s+\beta)}{\Gamma(s+1)\Gamma(s+\gamma)\Gamma(1-\alpha-s)}$$

$$= \frac{}{\Gamma(\alpha)\Gamma(\beta)\Gamma(1-\gamma)}\sum_{0}^{\infty}(-1)^s z^s \frac{\Gamma(s+\alpha)\Gamma(s+\beta)\Gamma(1-\gamma-s)}{\Gamma(s+1)}$$

$$= \frac{\Gamma(\gamma)\Gamma(1-\alpha)\Gamma(1-\beta)}{}\sum_{0}^{\infty}\frac{z^s}{\Gamma(s+1)\Gamma(s+\gamma)\Gamma(1-\alpha-s)\Gamma(1-\beta-s)}$$

etc

When $|z|<1$ and α unrestricted, that branch of the multiple-valued function $(1-z)^\alpha$ which has the value $+1$ when $z=0$ is represented by the binomial series

$$2)\quad (1-z)^\alpha = \frac{1}{\Gamma(-\alpha)}\sum_{s=0}^{\infty} z^s \frac{\Gamma(s-\alpha)}{\Gamma(s+1)} = \Gamma(\alpha+1)\sum_{0}^{\infty}\frac{(-1)^s z^s}{\Gamma(s+1)\,\Gamma(1+\alpha-s)} = \sum_{0}^{\infty} -1^s z^s C_{\alpha s}$$

$$\binom{\alpha}{s} = C_{\alpha s}$$

By term by term multiplication of the two series the first fundamental formula is found to be

$$3)\quad F(\alpha,\beta,\gamma;z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma-\alpha,\gamma-\beta,\gamma;z)$$

Whenever $F(\alpha,\beta,\gamma;1)$ is finite its value is

$$4)\quad F(\alpha,\beta,\gamma;1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)} \qquad (\text{Gauss})$$

Comparison of (1) and (2) shows that

$$5)\quad F(\alpha,\beta,\beta;z) = (1-z)^{-\alpha}$$

In eq (2), (3), (5) and in the following such as $(1)_a$ and (2) of II below, it is important to remember that $(1-z)^\alpha$ is the branch which is $+1$ when $z=0$; even when $z = 1-z_1^2$

Some elementary equivalents

6)$_a$ $z\,F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2) = \sin^{-1} z$

6)$_b$ $z\,F(\frac{1}{2}, 1, \frac{3}{2}; -z^2) = \tan^{-1} z$

if $|z| < 1$ (the branch which vanishes with

6)$_c$ $F(\frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{3}{2}; \sin^2\phi) = \frac{\sin \nu\phi}{\nu \sin\phi}$

6)$_d$ $F(1+\frac{\nu}{2}, 1-\frac{\nu}{2}, \frac{1}{2}; \sin^2\phi) = \frac{\sin \nu\phi}{\nu \sin\phi \cos\phi}$

6)$_e$ $F(\frac{\nu}{2}, -\frac{\nu}{2}, \frac{1}{2}; \sin^2\phi) = \cos \nu\phi$

6)$_f$ $F(\frac{1+\nu}{2}, \frac{1-\nu}{2}, \frac{1}{2}; \sin^2\phi) = \frac{\cos \nu\phi}{\cos\phi}$

$-\frac{\pi}{2} < \phi < \frac{\pi}{2}$

ν unrestricted (it may be complex)

6)$_g$ $F(1, 1, 2; z) = -\frac{1}{z} \log(1-z)$

6)$_h$ $2z\,F(\frac{1}{2}, 1, \frac{3}{2}; z^2) = \log \frac{1+z}{1-z}$

6)$_i$ $2\,F(-\frac{\nu}{2}, \frac{1-\nu}{2}, \frac{1}{2}; z^2) = (1+z)^\nu + (1-z)^\nu$

6)$_j$ $2\nu z\,F(1-\frac{\nu}{2}, \frac{1}{2}-\frac{\nu}{2}, \frac{3}{2}; z^2) = (1+z)^\nu - (1-z)^\nu$

$|z| < 1$

ν unrestricted

6)$_k$ $\frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1, \kappa^2) = K(\kappa)$ and $\frac{\pi}{2} F(-\frac{1}{2}, \frac{1}{2}, 1; \kappa^2) = E(\kappa)$

Use is made of the following Gamma formulas

7)$_a$ $z\,\Gamma(z) = \Gamma(z+1)$

7)$_b$ $\Gamma(z)\,\Gamma(1-z) = \dfrac{\pi}{\sin \pi z}$ $\qquad \dfrac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(n+1)} = \dfrac{1.3.5.\cdots(2n-1)}{2.4.6\cdots 2n}$

7)$_c$ $\Gamma(2z) = \dfrac{2^{2z}}{2\sqrt{\pi}}\,\Gamma(z)\,\Gamma(z+\tfrac{1}{2})$ $\qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \Gamma(-\tfrac{1}{2}) = -2\sqrt{\pi}$

If $|z|$ is large and $|\arg z| < \pi$

7)$_d$ $\Gamma(z) \approx \sqrt{2\pi}\, e^{-z} z^{\frac{1}{2}}(1+\epsilon) = \sqrt{2\pi}(1+\epsilon)\, e^{(z-\frac{1}{2})\log z}$ where $\epsilon = O(\frac{1}{z})$

7)$_e$ $\dfrac{\Gamma(z+\beta)}{\Gamma(z+\gamma)} \approx z^{\beta-\gamma}\left\{1 + \dfrac{(\beta-\gamma)(\beta+\gamma-\frac{1}{2})}{z} + O(\tfrac{1}{z^2})\right\}$ $\qquad |\arg z| < \pi$

8)$_a$ $\psi(z) \equiv \dfrac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \displaystyle\sum_{s=0}^{\infty}\left(\frac{1}{s+z} - \frac{1}{s+1}\right)$ where $\gamma = .57721566$ (Euler's const)

$\psi(1) = -\gamma$, $\psi(\frac{1}{2}) = -\gamma - \log$

8)$_b$ $\psi(1+z) - \psi(z) = \dfrac{1}{z}$ $\qquad \psi(n+\tfrac{1}{2}) = \psi(\tfrac{1}{2}) + \displaystyle\sum_{t=0}^{n-1\geq 0}\frac{1}{t+\frac{1}{2}}$

8)$_c$ $\psi(1-z) - \psi(z) = \pi \cot \pi z$ $\qquad \psi(n+1) = \psi(1) + \displaystyle\sum_{t=1}^{n}\frac{1}{t}$

8)$_d$ $2\psi(2z) = \psi(z) + \psi(z+\tfrac{1}{2}) + 2\log 2$

If B_n denote Bernoulli numbers

8)$_e$ $\psi(z) = \log z - \dfrac{1}{2z} + \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n B_n}{n z^{2n}}$ $\quad (B_1 = \frac{1}{6},\ B_2 = \frac{1}{30})$

8)$_f$ $\lim\limits_{\epsilon\to 0} \epsilon\,\Gamma(\epsilon - n) = \dfrac{(-1)^n}{\Gamma(1-n)}$ if n is a non-negative integer.

8)$_g$ $\dfrac{\psi(m)}{\Gamma(n)} = \dfrac{\infty}{\infty} = (-1)^{1-m}\,\Gamma(1-n)$ if m and n are both non-positive integers.

The functions $\Gamma(z)$ and $\Psi(z)$ are single-valued in the entire z-plane, their only singularities being simple poles where z is zero or a negative integer. Hence from the formula

9) $$\frac{1}{2\pi i}\oint_{t} \frac{f(\nu)\,d\nu}{(\nu-t)^{n+1}} = \frac{f^{(n)}(t)}{n!}$$ one obtains

9)$_a$ $$\frac{1}{2\pi i}\oint_{-\alpha-n} \Gamma(\nu+\alpha)\,d\nu = \frac{(-1)^n}{\Gamma(n+1)}$$

9)$_b$ $$\frac{1}{2\pi i}\oint_{n} \pi \cot \nu\pi \, d\nu = 1$$

Also for reference

10)$_a$ $$\int_0^z t^{x}(z-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} z^{x+y-1}$$

10)$_b$ $$\int_0^\infty t^{z-1} e^{-yt}\,dt = \frac{\Gamma(z)}{y^z} \quad \text{if } R(z) > 0$$

10)$_c$ $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+n}{2}+1\right)}$$

10)$_d$ $$\int_0^{\pi/2} \cos\nu\theta \cos^n\theta \, d\theta = \frac{\pi\, n!}{2^{n+1}\Gamma\left(\frac{n+\nu}{2}+1\right)\Gamma\left(\frac{n-\nu}{2}+1\right)}$$

Since this is not in general single valued in the unit circle $|z|=1$ a cut is necessary, which will be called the g-cut extending along the negative real axis from zero to $-\infty$, This g-cut restricts the range of arg z to $-\pi < \arg z < \pi$

Instead of using (13) it is more convenient (especially when γ is an integer) to take the two fundamental solutions of (11) as the functions f(z) and g(z) where for all values of z,

$$(14) \quad f(z) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;z) = \sum_{s=0}^{\infty} z^s \frac{\Gamma(s+\alpha)\Gamma(s+\beta)}{\Gamma(s+1)\Gamma(s+\gamma)}$$

$$= f(\alpha,\beta,\gamma;z)$$

and

$$(15) \quad g(\alpha,\beta,\gamma;z) \equiv -\pi\cot\gamma\pi\left[f(\alpha,\beta,\gamma;z) - z^{1-\gamma} f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)\right.$$

It will be noticed that f, F, and g are symmetrical functions of the first two parameters α,β. Also $F(\alpha,\beta,\gamma;z)$ is an integral function of α and β and a meromorphic function of γ, its only singularities as a single-valued function of γ are the simple poles when γ is a non-positive integer. By introducing the factor $\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}$ in the definition (14) of f, the lat

The hg diff. eq is

$$11)_a \quad z(1-z)y''(z) + [\gamma-(\alpha+\beta+1)z]\,y'(z) - \alpha\beta\, y(z) = 0$$

or

$$11)_b \quad D_z\left[z^{\gamma}(1-z)^{\alpha+\beta-\gamma+1}\, y'\right] = \alpha\beta\, z^{\gamma-1}(1-z)^{\alpha+\beta-\gamma}\, y$$

Letting $u = z^{p}(\ -z)^{q} y$ this becomes

$$12) \quad D_z\left[z^{\gamma-2p}(1-z)^{\alpha+\beta-\gamma+1-2q}\, u'\right] =$$

$$= u\cdot z^{\gamma-2p-1}(1-z)^{\alpha+\beta-\gamma-2q}\left[(p+q-\alpha)(p+q-\beta)+2p(\gamma-p)+\frac{p(p-\gamma-1)}{z}+\frac{q(\alpha+\beta-\gamma-q)}{1-z}\right]$$

The definition (1) of the hg function is equivalent to defining it as the solution of (11) which has the value $+1$ when $z=0$ and whose derivative y' has the value $\frac{\alpha\beta}{\gamma}$ when $z=0$. The expressions to be obtained below provide the analytic continuation of the hg function to the remainder of the plane outside the unit circle with center at the origin in which its definition (1) is valid. The plane must have an f-cut along the real axis of z from $+1$ to $+\infty$.

Another solution of (11) inside the circle $|z|=1$ is

$$13) \quad y(z) = z^{1-\gamma} F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; z)$$

Since this is not in general single valued inside the unit circle $|z|=1$ a cut is neccessary, which will be called the g-cut extending along the negative real axis from zero to $-\infty$, This g-cut restricts the range of $\arg z$ to $-\pi < \arg z < \pi$

Instead of using (13) it is more convenient (especially when γ is an integer) to take the two fundamental solutions of (11) as the functions $f(z)$ and $g(z)$ where for all values of z,

$$14) \quad f(z) \equiv \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;z) = \sum_{s=0}^{\infty} z^s \frac{\Gamma(s+\alpha)\Gamma(s+\beta)}{\Gamma(s+1)\Gamma(s+\gamma)}$$

$$= f(\alpha,\beta,\gamma;z)$$

and

$$15) \quad g(\alpha,\beta,\gamma;z) \equiv -\pi\cot\gamma\pi\left[f(\alpha,\beta,\gamma;z) - z^{1-\gamma} f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)\right]$$

It will be noticed that f, F, and g are symmetrical functions of the first two parameters α,β. Also by (1) $F(\alpha,\beta,\gamma;z)$ is an integral function of α and β and a meromorphic function of γ, its only singularities as a single-valued function of γ are the simple poles when γ is a non-positive integer. By introducing the factor $\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}$ in the definition (14) of f, the latter

becomes an integral function of γ, but as a function of α or β it has simple poles at the non-positive integers. Or $\frac{F(\alpha,\beta,\gamma;z)}{\Gamma(\gamma)}$ is an integral function of every parameter.

The two functions and their derivatives are connected by the relations

16)$_a$ $f'(\alpha,\beta,\gamma;z) = f(\alpha+1,\beta+1,\gamma+1;z)$

16)$_b$ $g'(\alpha,\beta,\gamma;z) = g(\alpha+1,\beta+1,\gamma+1;z)$

It is readily found that

17) $f(\alpha,\beta,\gamma;z)\,g'(\alpha,\beta,\gamma;z) - f'(\alpha,\beta,\gamma;z)\,g(\alpha,\beta,\gamma;z) =$

$$= z^{-\gamma}(1-z)^{\gamma-\alpha-\beta-1}\cos\gamma\pi\,\Gamma(\alpha)\Gamma(\alpha+1-\gamma)\Gamma(\beta)\Gamma(\beta+1-\gamma)$$

which shows for what values of the parameters f and g are linearly independant solutions of (11).

The definition (14) together with (1) shows that when $\gamma \to n$ where n is any real integer

18)$_a$ $f(\alpha,\beta,n;z) = z^{1-n} f(\alpha+1-n,\beta+1-n,2-n;z)$

On the other hand the definition (15) shows that g satisfies this same functional equation identically, that is for all values of α,β,γ,

18)$_b$ $g(\alpha,\beta,\gamma;z) \equiv z^{1-\gamma} g(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z)$.

This formula is useful when γ is an integer, for

it gives immediately a g function in which the third parameter is a positive integer.

When $\gamma \to n$ where n is an integer the factor $\cot \gamma\pi$ becomes infinite but g being an integral function of γ does not, for by $(18)_a$ the bracket in (15) then $\to 0$. Evaluating the "indeterminate" form $\infty \cdot 0$ by use of (7) it is found that if $n = 0, 1, 2, 3, ---$

19) $$g(\alpha, \beta, n; z) = -f(\alpha, \beta, n; z) \log z$$

$$+ \sum_{s=-1}^{s=-(n-1)<0} (-1)^s z^s \frac{\Gamma(-s)\,\Gamma(s+\alpha)\,\Gamma(s+\beta)}{\Gamma(s+n)}$$

$$-\sum_{s=0}^{\infty} z^s \frac{\Gamma(s+\alpha)\,\Gamma(s+\beta)}{\Gamma(s+1)\,\Gamma(s+n)}\Big[\psi(s+\alpha) + \psi(s+\beta) - \psi(s+n) - \psi(s+1)\Big]$$

where $\log z$ has its principal value ($-\pi < \arg z < \pi$) and where the sum of negative powers of z is absent in the cases $n=0$ and $n=1$. The factor of $\log z$ could be replaced by the second member of $(18)_a$ — a procedure desirable in the case $n=0$.

At the g-cut where $z = x \pm i0$, $x < 0$ the eqn (15) shows that

20) $$g(\alpha,\beta,\gamma;x+i0) - g(\alpha,\beta,\gamma;x-i0) = 2\pi i \cos\gamma\pi\,|x|^{1-\gamma} f(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;x)$$

this difference being zero if $x > 0$ where there is no g-cut.

The equations given thus far apply only when $|z|<1$ but it will be found that this relations (20) like (15) (16)$_a$ 16)$_g$ (17) 18)$_a$ and (18)$_g$ is a functional relation holding everywhere.

It must be remembered that the f-cut from $+1$ to $+\infty$ is not a cut for $g(z)$ nor for such functions as $\log z$, z^ν, or $(z+1)^\nu$. The g-cut from zero to $-\infty$ along the real axis is necessary not only for $g(z)$ but also for z^ν and $\log z$, but not for $f(z)$, the neighborhood of the real axis for which $x<1$ consists of ordinary points for the function $f(z)$. In making analytic continuations of $f(z)$ and $g(z)$ to regions of the suitably cut plane outside the circle $|z|=1$ there appears the multiple-valued function $(1-z)^\nu$ and $\log(1-z)$. These must be understood to represent principal values so that $-\pi < \arg(1-z) < \pi$. For such functions the f-cut suffices, if the value of $\arg(1-z)$ is $-\pi$ just above and $+\pi$ just below the f-cut.

Although the f- and g-cuts alone suffice for the continuation of $f(z)$ and $g(z)$ thus leaving open the part of the real axis between $z=0$ and $z=+1$ it will be found convenient for practical applications to express such terms as $(1-z)^\nu$ or $\log(1-z)$ in terms of $z-1$ and $\log(z-1)$ respectively. The principal value of $\arg(z-1)$ lies between $-\pi$ and $+\pi$ being zero on both sides of the f-cut which is not a cut for $(z-1)^\nu$ or $\log(z-1)$. For these functions the real axis from $+1$ to $-\infty$ is a cut, and $\arg(z-1)$ is $+\pi$ just above it, $-\pi$ just below it.

Hence in the following

21) $$\left\{\begin{aligned} \arg(z-1) &= \arg(1-z) \pm \pi \\ (z-1)^\nu &= (1-z)^\nu e^{\pm i\nu\pi} \\ \log(z-1) &= \log(1-z) \pm i\pi \end{aligned}\right\}$$

The upper sign always applies when $y>0$ the lower when $y<0$

In the formulas below where $e^{\pm i\nu\pi}$ or $e^{\mp i\nu\pi}$ occur the upper or lower sign corresponds to z in the upper or lower half-plane respectively.

II Homographic Substitutions

The hypergeometric differential equation (11) I is of Fuchsian type with three regular singular points $z = 0, 1, \infty$. The homographic substitutions which interchange the three singular points among themselves lead to a differential eq'n in general of type (12) I from which a new differential equation of hypergeometric type with new parameters as functions of the original ones is recovered by change of dependent variable as in passing from (12) to (11). The fundamental solutions will be a new pair of $f(z)$ and $g(z)$ functions of the new variable z' so that the $f(z)$ and $g(z)$ will be (save for factors introduced by the transformation) linear functions of the new ones. The coefficients of these linear relations may be found by letting $z=0$ and $z=1$ using (4) I.*

The six transformations of type $z' = \dfrac{Az+B}{Cz+D}$ which interchange two singular points leaving the third unaltered are (including the identical transformation)

$$z' = z, \quad z' = 1-z, \quad z' = \frac{z}{z-1}, \quad z' = \frac{z-1}{z}, \quad z' = \frac{1}{1-z}, \quad z' = \frac{1}{z}.$$

* The results obtained for parameters making the series converge at $z=1$, continue valid for all values of the parameters since they are analytic functions of the parameters.

Gauss's transformation of the hg function is obtained by making the substitution $z' = 1-z$ in (11) which transforms into a hg diff eq with parameters $\alpha,\ \beta,\ 1-\gamma+\alpha+\beta$, independent variable z'.

Hence $F(\alpha,\beta,\gamma;z)$ is a linear function of any two fundamental solutions of the new equation, in any region which is a common domain of existence for the three which in this case is the area of the z-plane common to the two circles $|z|=1$ and $|z-1|=1$ The coefficients are determined by letting $z\to 0$ and by letting $z\to 1$ using (4) I.

This determines the following known as Gauss's transformation

$$1)_a \quad F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\alpha,\beta,1-\gamma+\alpha+\beta;1-z)$$

$$+(1-z)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\ \Gamma(\beta)} F(\gamma-\alpha,\gamma-\beta,1+\gamma-\alpha-\beta;1-z)$$

which is valid wherever the second member has a meaning i,e inside the circle $|z-1|=1$. The f-cut is neccesary because of the factor $(1-z)^{\gamma-\alpha-\beta}$

This may also be written

$$1)_b \quad f(\alpha,\beta,\gamma;z) = \frac{g(\alpha,\beta,1-\gamma+\alpha+\beta;1-z)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\cos(\gamma-\alpha-\beta)\pi}$$

$$=(1-z)^{\gamma-\alpha-\beta}\frac{g(\gamma-\alpha,\gamma-\beta,1+\gamma-\alpha-\beta;1-z)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\cos(\gamma-\alpha-\beta)\pi}$$

When $\gamma-\alpha-\beta$ is an integer one of these expressions becomes a g-function of $1-z$ whose third parameter is a positive integer and hence it is given by (19) of I

By an obvious change of notation this may be written

$$1)_c \quad g(\alpha,\beta,\gamma;z) \equiv z^{1-\gamma} g(\alpha+1-\gamma,\beta+1-\gamma,2-\gamma;z) =$$

$$= -\cos\gamma\pi\,\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)\,f(\alpha,\beta,1-\gamma+\alpha+\beta;1-z)$$

which shows that $g(z)$ is just another f-function of $1-z$ multiplied by a factor independant of z.

This shows why the f cut in the z' plane makes the g cut for $g(z)$ in the z plane from 0 to $-\infty$ since this corresponds to z' from 1 to $+\infty$.

It may be noticed that for every transformation of $F(z)$ or $f(z)$ we obtain one for $g(z)$ either by the original definition of g in (15) I or by use of $1)_c$

Euler's transformation is obtained by the substitution $z' = \frac{z}{z-1}$ in (11) I. It gives

$$2)\quad F(\alpha,\beta,\gamma;z) = (1-z)^{-\beta} F(\gamma-\alpha,\beta,\gamma;\tfrac{z}{z-1})$$

$$= (1-z)^{-\alpha} F(\alpha,\gamma-\beta,\gamma;\tfrac{z}{z-1})$$

which continues $F(z)$ to the half-plane $R(z) < \frac{1}{2}$ where it is single valued, since $\arg(1-z)$ is continuous in the neighborhood of the real axis for which $x<1$. Two applications of (2) give (3) I. The $g(z)$ function is transformed into functions of $\frac{z}{z-1}$ by the definition (15) I. A special case $z=-1$ of (2) is

$$2')\quad F(\alpha,\beta,\gamma;-1) = 2^{-\beta} F(\gamma-\alpha,\beta,\gamma;\tfrac{1}{2})$$

The continuation of $F(\alpha,\beta,\gamma,z)$ to the other half plane where $R(z) > \frac{1}{2}$ that is $\left|\frac{z-1}{z}\right| < 1$ is furnished by the substitution $z' = \frac{z-1}{z}$ in (11) I. This gives the same result as applying Euler's transformation to each of the F functions in the second member of Gauss's transformation 1)a. The result is

$3)_a$ $$F(\alpha,\beta,\gamma,z) = z^{-\beta}\left[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} F(\beta+1-\gamma,\beta,1-\gamma+\alpha+\beta, \tfrac{z-1}{z}) + \left(\frac{1-z}{z}\right)^{\gamma-\alpha-\beta}\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} F(\gamma-\alpha,1-\alpha,1+\gamma-\alpha-\beta; \tfrac{z-1}{z})\right]$$

$\mathcal{R}(z) > \frac{1}{2}$

where α and β could be interchanged.
This shows $F(\alpha\,\beta\,\gamma\,z)$ varies continuously when z crosses the real axis between $\frac{1}{2}$ and 1 where there is no "f-cut".
For the case where $\gamma-\alpha-\beta$ is an integer this becomes indeterminate, and the following equivalent form is more convenient

$3)_b$ $$f(\alpha,\beta\,\gamma;z) = \frac{z^{-\beta}\Gamma(\alpha)}{\Gamma(\gamma-\alpha)}\left[e^{\pm i\alpha\pi} f(\beta+1-\gamma,\beta,1-\gamma+\alpha+\beta\cdot \tfrac{z-1}{z}) + \frac{\sin\alpha\pi\, e^{\mp i(\gamma-\alpha-\beta)\pi}}{\pi\cos(\gamma-\alpha-\beta)\pi} g(\beta+1-\gamma,\beta,1-\gamma+\alpha+\beta; \tfrac{z-1}{z})\right]$$

$\mathcal{R}(z) > \frac{1}{2}$

where as in (21) I the upper or lower sign is to be taken according as z is in the upper or lower half-plane. This shows that

$3)_c$ $$f(\alpha,\beta\,\gamma\, x+i0) - f(\alpha,\beta,\gamma;x-i0) = 2\pi i\, x^{-\beta}\left(\frac{x-1}{x}\right)^{\gamma-\alpha-\beta} F(\gamma-\alpha,1-\alpha,1+\gamma-\alpha-\beta;\tfrac{x-1}{x}) \text{ if } x>1$$
$$= 0 \text{ if } x<1$$

The eqn(2) and (3) give the analytic expression for $F(\alpha,\beta,\gamma;z)$ in the entire "f-cut" z-plane.

The continuation of $F(\alpha,\beta,\gamma;z)$ to all points in the cut, z-plane which are outside the circle $|z|=1$ is furnished by the substitution $z'=\frac{1}{z}$ in (11) I. The result is the same as obtained by applying Gauss's transformation $(1)_a$ to each of the F-functions in the second member of $(3)_a$

$$4)_a \quad F(\alpha,\beta,\gamma;z)=\frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\beta)}\,e^{\pm i\alpha\pi}z^{-\alpha}F\left(\alpha,\alpha+1-\gamma,1+\alpha-\beta;\frac{1}{z}\right)$$

$$+\frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\,\Gamma(\alpha)}\,e^{\pm i\beta\pi}z^{-\beta}F\left(\beta,\beta+1-\gamma,1-\alpha+\beta;\frac{1}{z}\right)$$

$$|z|>1$$

For cases where $\alpha-\beta$ is integral, this is put in the form

$$4)_b \quad f(\alpha,\beta,\gamma;z)=z^{-\alpha}\left[e^{\mp i(\gamma-\alpha-\beta)\pi}f\left(\alpha,\alpha+1-\gamma,1+\alpha-\beta;\frac{1}{z}\right)\right.$$

$$\left.+\frac{e^{\pm i\beta\pi}\sin(\gamma-\beta)\pi}{\pi\cos(\alpha-\beta)\pi}\,g\left(\alpha,\alpha+1-\gamma,1+\alpha-\beta;\frac{1}{z}\right)\right]$$

$$|z|>1$$

upper or lower sign according as z is in upper or lower half-plane.

Eqns 4 together with (1) I cover the entire plane.

Applying Euler's transformation (2) to each of the F functions in the second member of (4)a gives

$$5)_a \quad F(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)\,\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\beta)}(1-z)^{-\alpha} F\left(\alpha, \gamma-\beta, 1+\alpha-\beta; \frac{1}{1-z}\right)$$

$$+ \frac{\Gamma(\gamma)\,\Gamma(\alpha-\beta)}{\Gamma(\gamma-\beta)\,\Gamma(\alpha)}(1-z)^{-\beta} F\left(\gamma-\alpha, \beta, 1-\alpha+\beta; \frac{1}{1-z}\right)$$

that is

$$5)_e \quad f(\alpha,\beta,\gamma;z) = (1-z)^{-\alpha} \frac{g\left(\alpha, \gamma-\beta, 1+\alpha-\beta; \frac{1}{1-z}\right)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\cos(\gamma-\alpha-\beta)\pi}$$

$$\equiv (1-z)^{-\beta} \frac{g\left(\gamma-\alpha, \beta, 1-\alpha+\beta; \frac{1}{1-z}\right)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)\cos(\gamma-\alpha-\beta)\pi}$$

These with (1) cover the entire plane.

The homographic substitutions therefore cover the plane in three different ways.

III Non-Linear Substitutions

In the hg eq (11) of I change the independant variable from z to ω by the substitution

1)$_a$ $$\omega = \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}$$ which is equivalent to

1)$_b$ $$z = \frac{-4\omega}{(\omega-1)^2}$$

The explanation of cuts and of arg(1-z) in section I between equ(20) and(21) shows that 1)$_a$ uniquely specifies the branch and 1)$_b$ is its unique equivalent. The relation 1)$_a$ or 1)$_b$ represents conformally the entire z-plane upon the interior of the circle $|\omega|=1$ in the ω-plane, the f-cut being the perimeter. This is indicated by similar lettering in figures 1 and 2b. The transform of (11) I is

2) $$D_\omega\left[\omega^{\gamma}(1-\omega)^{1-2\alpha-2\beta}\cdot(1+\omega)^{1-2(\gamma-\alpha-\beta)}\cdot D_\omega y\right]+4\alpha\beta\,\omega^{\gamma-1}(1-\omega)^{-1-2\alpha-2\beta}(1+\omega)^{1-2(\gamma-\alpha-\beta)}y=0$$

Setting

3) $y=(1-\omega)^{2\alpha}u$ this reduces to a Fuchsian equ. whose four regular singular points are $\omega_0=0, \omega_1=1, \omega_2=-1, \omega_3=\infty$.

4) $$D_\omega^2 u+PD_\omega u+Qu=0 \text{ where } \begin{cases} P=\dfrac{\gamma}{\omega}+\dfrac{1+2\alpha-2\beta}{\omega-1}+\dfrac{1-2(\gamma-\alpha-\beta)}{\omega+1} \\[2ex] Q=\dfrac{2\alpha\left[(2\alpha+1-\gamma)\omega+\gamma-2\beta\right]}{\omega(\omega-1)(\omega+1)} \end{cases}$$

It is proven in treatises on differential equations that the two fundamental integrals of (4) for the neighborhood of $\omega = 0$ are given by series of ascending powers of ω which converge inside any circle with center at the origin which does not contain any other singular point. In this case the circle of convergence is $|\omega| = 1$. Hence the function

4)$_a$ $$u = (1-\omega)^{-2\alpha} F(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} C_n \omega^n \equiv F_1(\omega) \quad \text{if } |\omega| < 1$$

is that solution of (4) which satisfies the initial conditions,

$$u = 1 \quad \text{and} \quad \frac{du}{d\omega} = 2\alpha\left(1 - \frac{2\beta}{\gamma}\right) \quad \text{when } \omega = 0$$

Equ(4) requires the coefficients to satisfy the difference equation

5) $$(n+2)(n+1+\gamma)C_{n+2} = 2(\gamma-2\beta)(n+1+\alpha)C_{n+1} + (n+2\alpha)(n+2\alpha+1-\gamma)C_n$$

The initial conditions are

5)$_a$ $$C_0 = 1, \quad C_1 = \frac{2\alpha(\gamma-2\beta)}{\gamma}$$

The solution is

6) $$C_n = \frac{2\sqrt{\pi}\,\Gamma(\gamma)}{2^{2\alpha}\Gamma(\alpha)\Gamma(\beta)} \sum_{s=0}^{n} (-1)^s \frac{\Gamma(s+2\alpha+n)\,\Gamma(s+\beta)}{\Gamma(s+1)\,\Gamma(s+\gamma)\,\Gamma(s+\alpha+\frac{1}{2})\,\Gamma(1+n-s)}$$

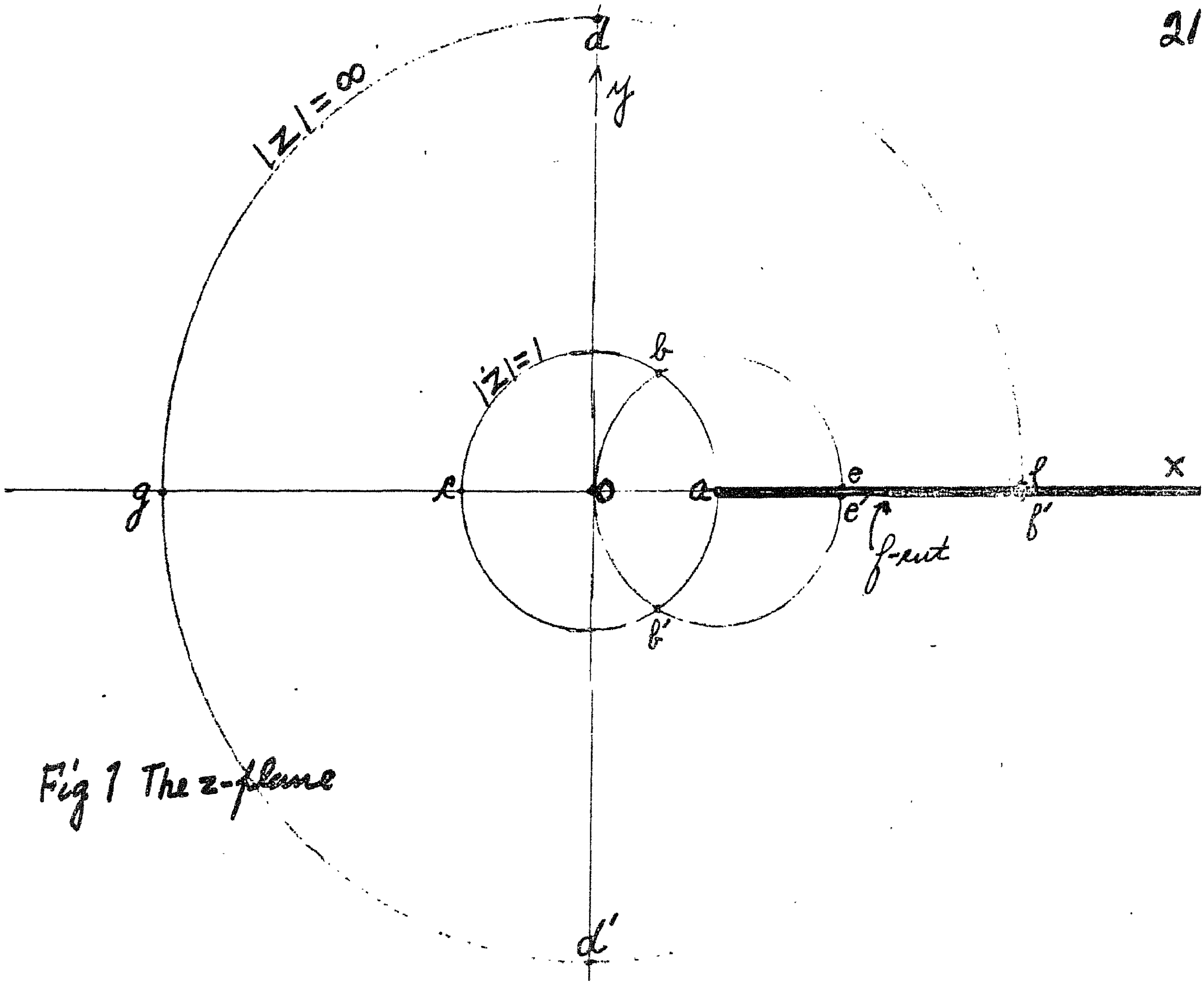


Fig 1 The z-plane

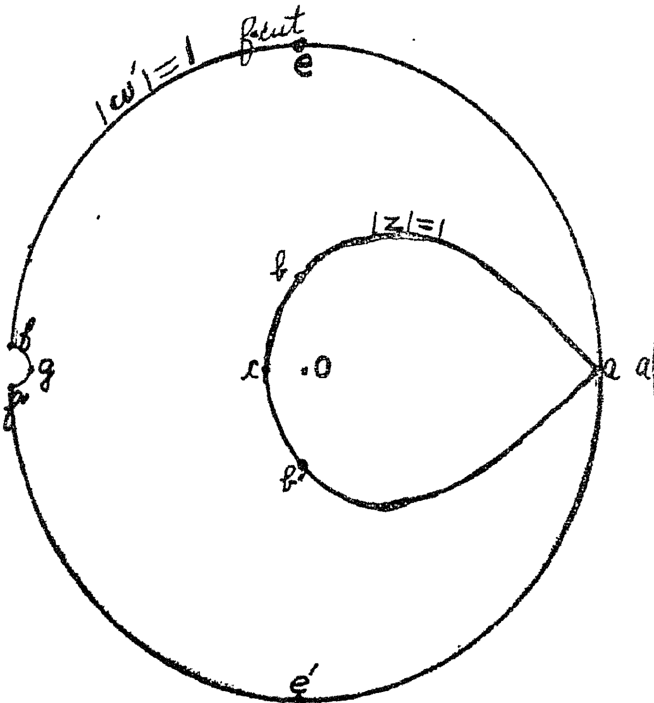


Fig 2a The w′ circle $w' = \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}$

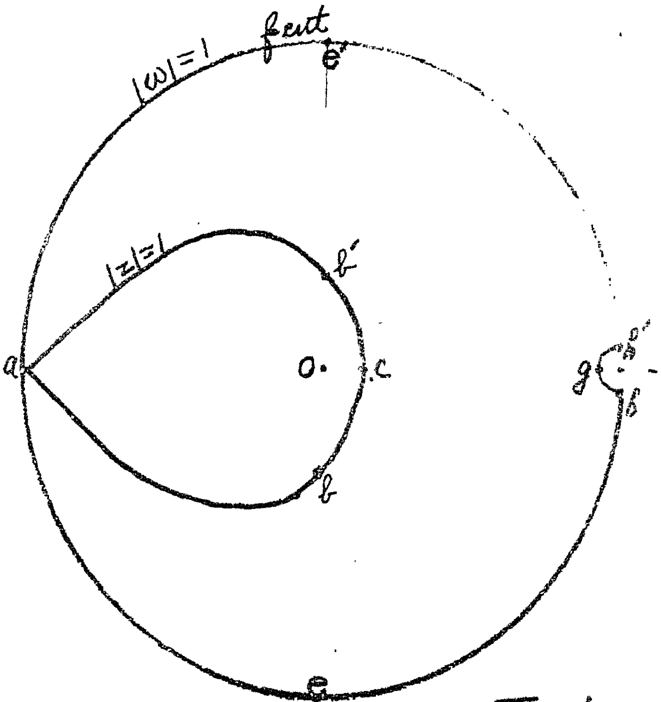


Fig 2b The w circle $w = \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}$

This is obtained by expressing z in terms of ω and expanding the left side of $(4)_a$ in powers of ω, and then equating coefficients of like powers of ω on both sides of the equation.

The series

$$7)\quad f(\alpha,\beta,\gamma;z) = \frac{(1-\omega)^{2\alpha}\,2\sqrt{\pi}}{2^{2\alpha}}\sum_{n=0}^{\infty}\omega^{n}\sum_{s=0}^{n}(-1)^{s}\frac{\Gamma(s+2\alpha+n)\,\Gamma(s+\beta)}{\Gamma(s+1)\,\Gamma(s+\gamma)\,\Gamma(s+\alpha+\frac{1}{2})\,\Gamma(1+n-s)}$$

is therefore valid for all values of z in the plane with an f cut where $\omega = \dfrac{\sqrt{1-z}-1}{\sqrt{1-z}+1}$.

In the special case where $\gamma = \alpha+\beta+\frac{1}{2}$ the eq (2) is a hg. diff eq and (7) becomes

$$8)_a\quad F(\alpha,\beta,\alpha+\beta+\tfrac{1}{2};z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha,\alpha-\beta+\tfrac{1}{2},\alpha+\beta+\tfrac{1}{2};\frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

for all values of z

that is

$$8)_b\quad F(2\alpha,\alpha-\beta+\tfrac{1}{2},\alpha+\beta+\tfrac{1}{2};\omega) = (1-\omega)^{-2\alpha}\,F\left(\alpha,\beta,\alpha+\beta+\tfrac{1}{2};\frac{-4\omega}{(\omega-1)^2}\right)$$

which is valid when ω is inside the locus $|z|=1$ of fig 2b.

Applying Euler's transformation to the second member of this gives

$$9)_a\quad F(2\alpha,\alpha-\beta+\tfrac{1}{2},\alpha+\beta+\tfrac{1}{2};\omega') = (1+\omega')^{-2\alpha}\,F\left(\alpha,\alpha+\tfrac{1}{2},\alpha+\beta+\tfrac{1}{2};\frac{4\omega'}{(1+\omega')^2}\right)$$

where ω' has been written for ω. The conditions under which this is valid are seen by making the substitution

$1)'\quad z = \dfrac{4\omega'}{(1+\omega')^2}$ so that $\omega' = \dfrac{1-\sqrt{1-z}}{1+\sqrt{1-z}} = -\omega$ by (1). Hence

eq 8c is valid when ω' is inside the closed curve of fig 2a which is the locus of $|z|=1$, since this figure shows how the "f-cut" z-plane is conformally represented upon the interior of the circle $|\omega'|=1$ by the relation (1)'. The equ 9)a may then be written

$$9)_b \quad F\left(\alpha, \alpha+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}; z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, \alpha-\beta+\tfrac{1}{2}, \alpha+\beta+\tfrac{1}{2}; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

for all values of z (with an f-cut)

Eq 9)a and (9)b may be regarded as special cases of the two following

In the case $\beta=\alpha+\frac{1}{2}$ eq (2) is a hg. diff. eq. with independant variable $\omega'=-\omega$ and (7) becomes

$$10)_a \quad F(\alpha, \alpha+\tfrac{1}{2}, \gamma; z) = (1+\omega')^{2\alpha} F(2\alpha, 2\alpha+1-\gamma, \gamma; \omega')$$

$$= \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, 2\alpha+1-\gamma, \gamma; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

valid for the entire z-plane.

This may be written

$$10)_b \quad F(2\alpha, 2\alpha+1-\gamma, \gamma; \omega') = (1+\omega')^{-2\alpha} F\left(\alpha, \alpha+\tfrac{1}{2}, \gamma; \frac{4\omega'}{(1+\omega')^2}\right)$$

which is valid when ω' is inside the curve $|z|=1$ of fig 2a

Transforming the second member of this eqn by Euler's transformation gives after dropping the prime from ω'

$$11)_a \quad F(2\alpha, 2\alpha+1-\gamma, \gamma; \omega) = (1-\omega)^{-2\alpha} F\left(\alpha, \gamma-\alpha+\tfrac{1}{2}, \gamma; \frac{-4\omega}{(\omega-1)^2}\right)$$

which is valid under the same conditions as $(8)_b$

This may be written

$$11)_b \quad F(\alpha, \gamma-\alpha+\tfrac{1}{2}, \gamma; z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, 2\alpha+1-\gamma, \gamma; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

valid for all values of z

Application of Euler's theorem to the second member of eq $8)_a$ gives

$$12)_a \quad F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}; z) = F\left(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}; \frac{1-\sqrt{1-z}}{2}\right)$$

which may be written. if $\left|\frac{-\sqrt{1-z}}{2}\right|<1$

$$12)_b \quad F(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}; z_1) = F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}; 4z_1(1-z_1))$$

if $|4z_1(1-z_1)|<1$

where

$$12)_c \quad z_1 = \frac{1-\sqrt{1-z}}{2} \quad \text{or} \quad z = 4z_1(1-z_1)$$

The z-plane is represented on the half of the z_1-plane for which $x_1 < \frac{1}{2}$ this line $x = \frac{1}{2}$ represents the point of z-plane

Equ$(12)_a$ is valid if z is inside the cardiod in the z-plane fig 3a which corresponds to z_1 inside that part of the circle $|z_1| = 1$ of the z_1-plane fig 3b for which $x_1 < \frac{1}{2}$. Therefore in equation $(12)_a$ the

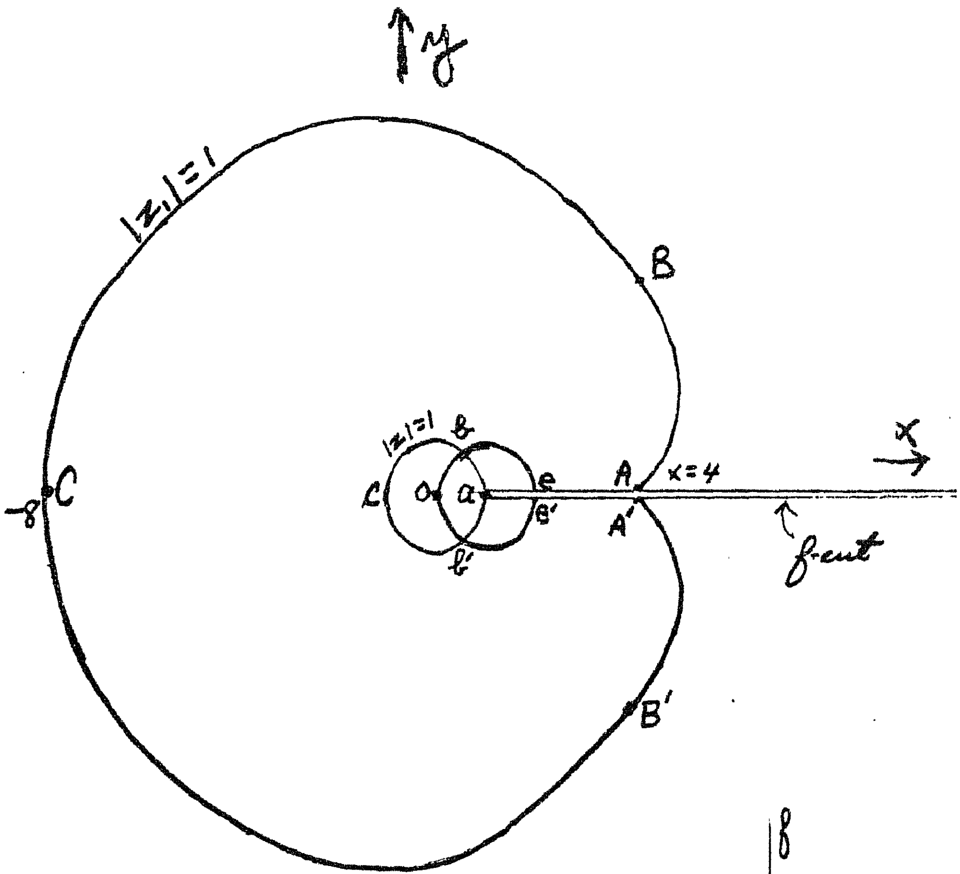
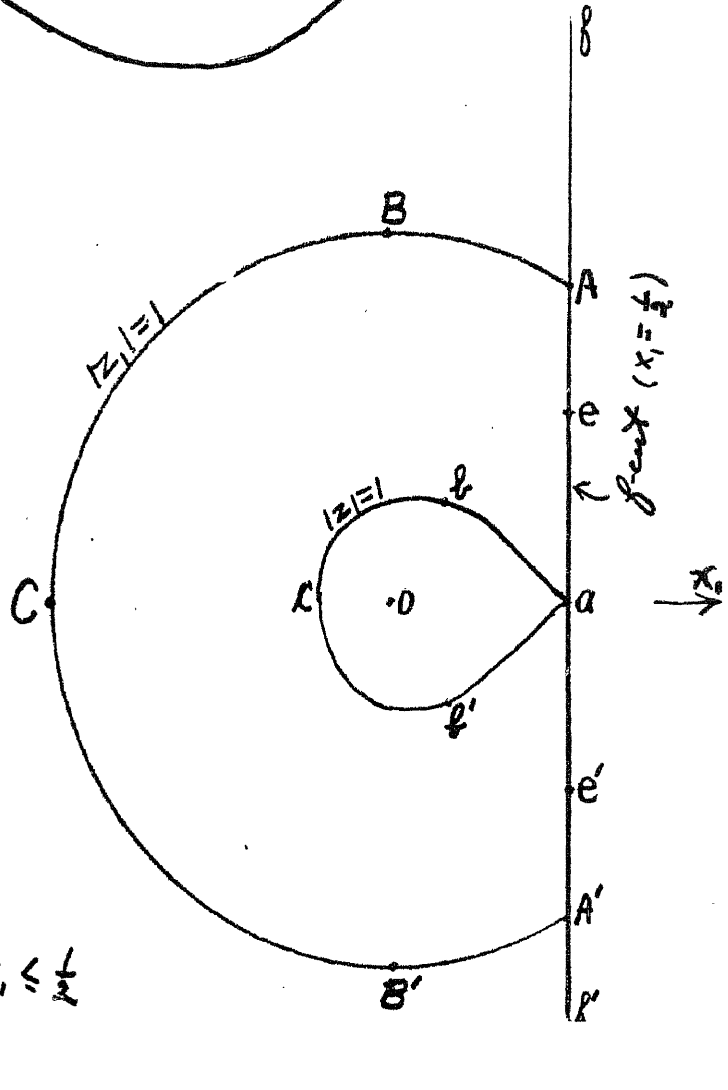


Fig 3a The z-plane

$z = 4z_1(1-z_1)$



$z_1 = \underline{1-\sqrt{1-z}}$

Fig 3b The z_1 half-plane $x_1 \le \frac{1}{2}$

second member gives a considerable analytic continuation of the function on the left side. In eqn 12)$_b$ this is inverted the second member only has a meaning when z lies in the lobe of the lemniscate where $x_1 < \frac{1}{2}$ and $|4z_1(z_1-1)| < 1$ fig 3$_b$.

The transformation of $(12)_a$ gives

$$13)_a \quad F\left(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}, 1-z)$$

or by Gauss's theorem

$$13)_b \quad F\left(2\alpha, 2\beta, \alpha+\beta+\tfrac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})} F(\alpha, \beta, \tfrac{1}{2}, z)$$

$$- z^{\frac{1}{2}} \frac{2\sqrt{\pi}\,\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha)\,\Gamma(\beta)} F(\alpha+\tfrac{1}{2}, \beta+\tfrac{1}{2}, \tfrac{3}{2}; z)$$

This corresponds to

$13)_c \quad z_2 = \dfrac{1-\sqrt{z}}{2}$ or $z = (2z_2 - 1)^2$ which represents the entire

z-plane with a g-cut upon the half plane $x_2 < \frac{1}{2}$ the g-cut corresponding in this case to the line $x_2 = \frac{1}{2}$. The conformation is shown by similar lettering in figures $4)_a$ and $4)_b$. The first member of (13) provides the analytic continuation of the second

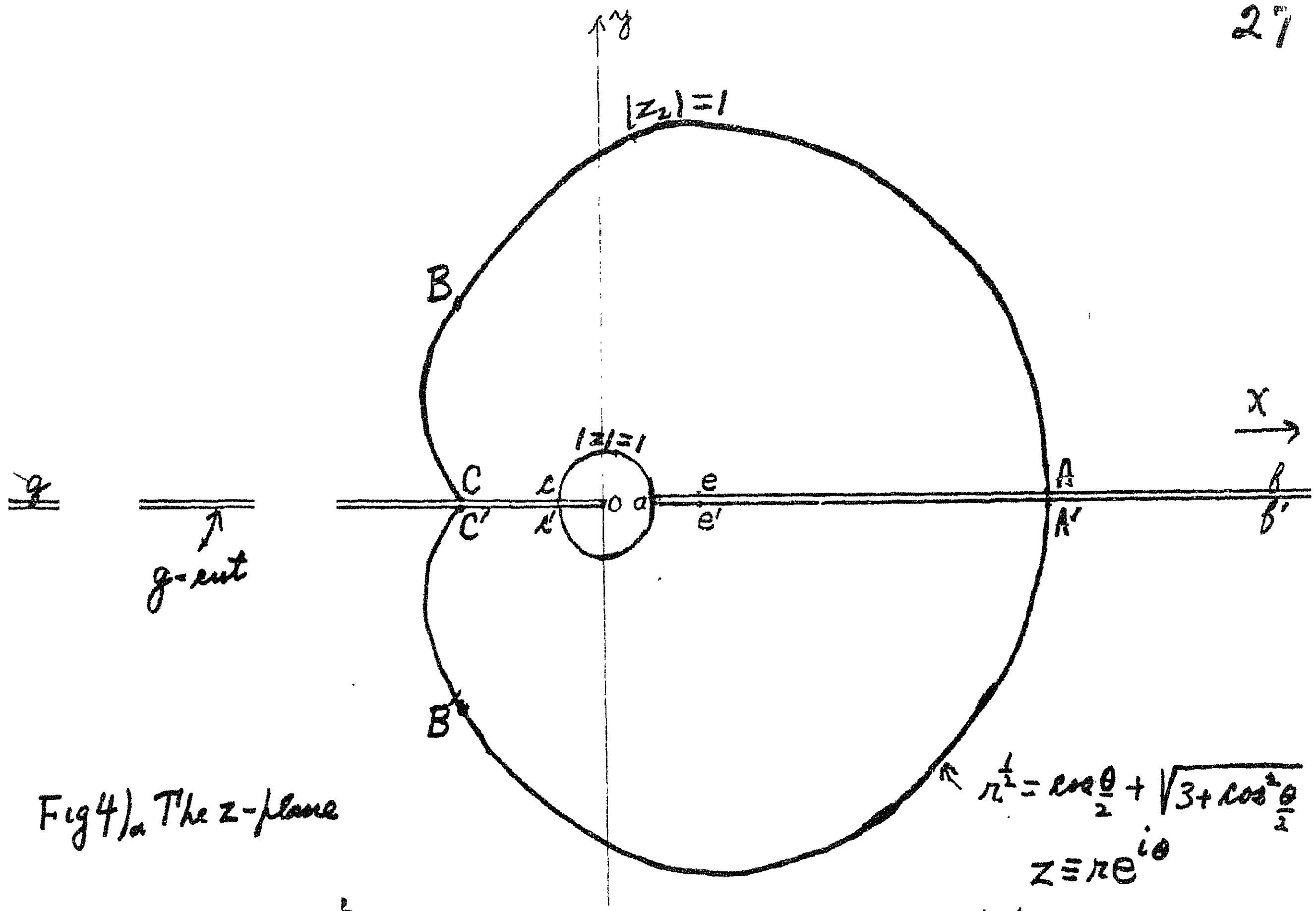


Fig 4)a The z-plane

$Z = (2Z_2-$

$Z_2 = \frac{1-\sqrt{Z}}{2}$

$r_2 \leq \frac{1}{2}$

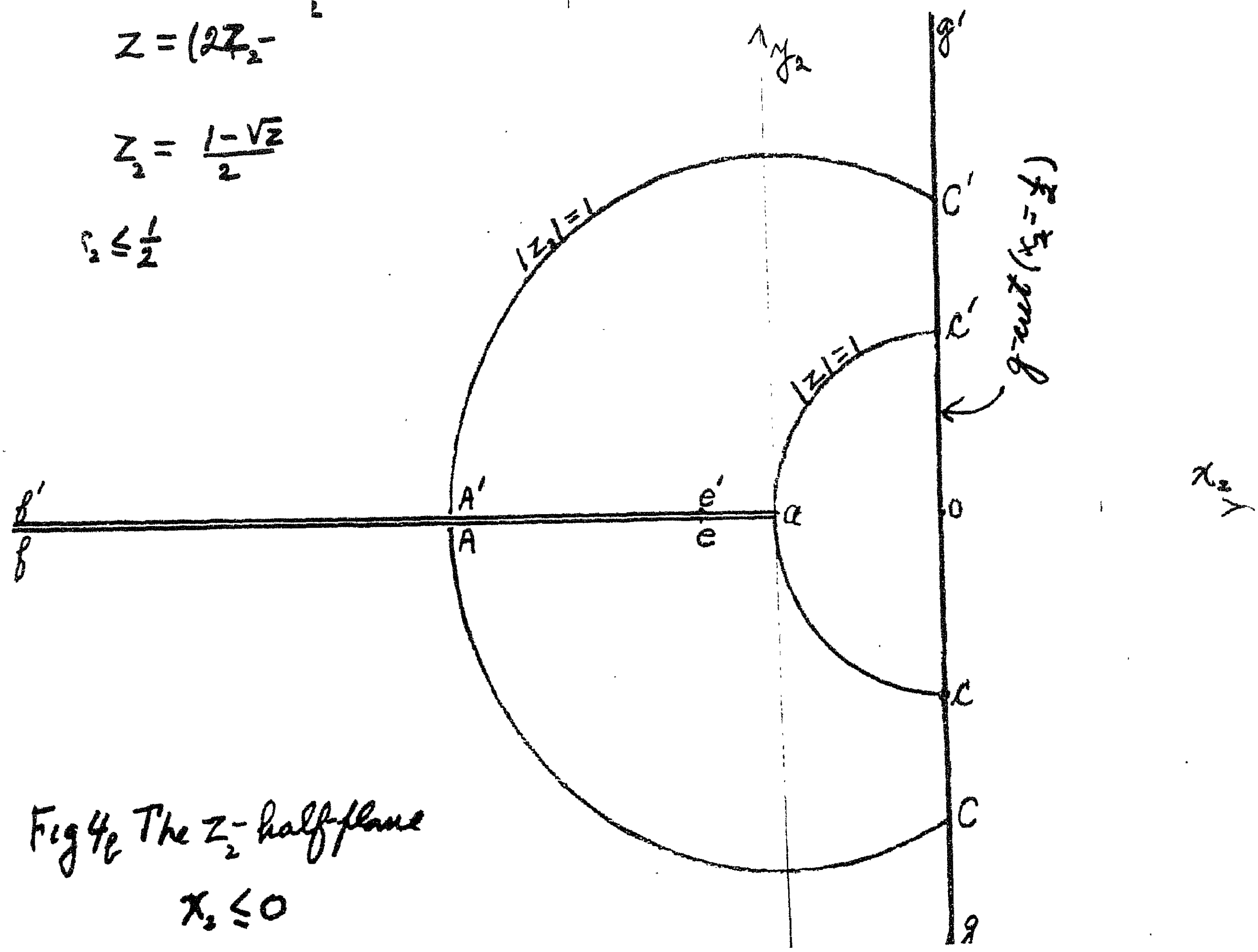


Fig 4b The Z_2-half-plane
$X_2 \leq 0$

member to the interior of the cardiod of fig 4)$_a$ which corresponds in fig 4)$_b$ to that part of the interior of the circle $|z_2|=1$ for which $x_2 < \frac{1}{2}$

The eq(13)$_b$ is useful in transforming associated Legendre functions. It is valid when z is inside the circle of the z-plane $|z|=1$ with the g-cut coc' of fig 4$_a$. This region corresponds to the semicircle of the z_2 plane $|z_2-\frac{1}{2}|=\frac{1}{2}$, $x_2<\frac{1}{2}$ in fig(4)$_b$.

The special case $z=0$ of (13)$_b$ gives

$$14)_a \quad F\left(\alpha,\beta,\frac{\alpha+\beta+1}{2};\frac{1}{2}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{\alpha+\beta+1}{2}\right)}{\Gamma\left(\frac{\alpha+1}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}$$

or by Euler's transformation

$$14)_b \quad F(\alpha,\beta,1-\alpha+\beta;-1)=\frac{2^{-\beta}\sqrt{\pi}\,\Gamma(1-\alpha+\beta)}{\Gamma\left(1-\alpha+\frac{\beta}{2}\right)\Gamma\left(\frac{\beta+1}{2}\right)}=\frac{2^{-\alpha}\sqrt{\pi}\,\Gamma(1+\alpha-\beta)}{\Gamma\left(1-\beta+\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+1}{2}\right)}$$

A third special case of (7) gives Kummer's transformation. This is when $\gamma = 2\beta$ in which case eq 4 becomes a hypergeometric differential equation. Equation 7 then becomes Kummer's transformation

$$15)_a \quad F(\alpha, \beta, 2\beta; z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(\alpha, \alpha-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}; \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right)$$

which may be written

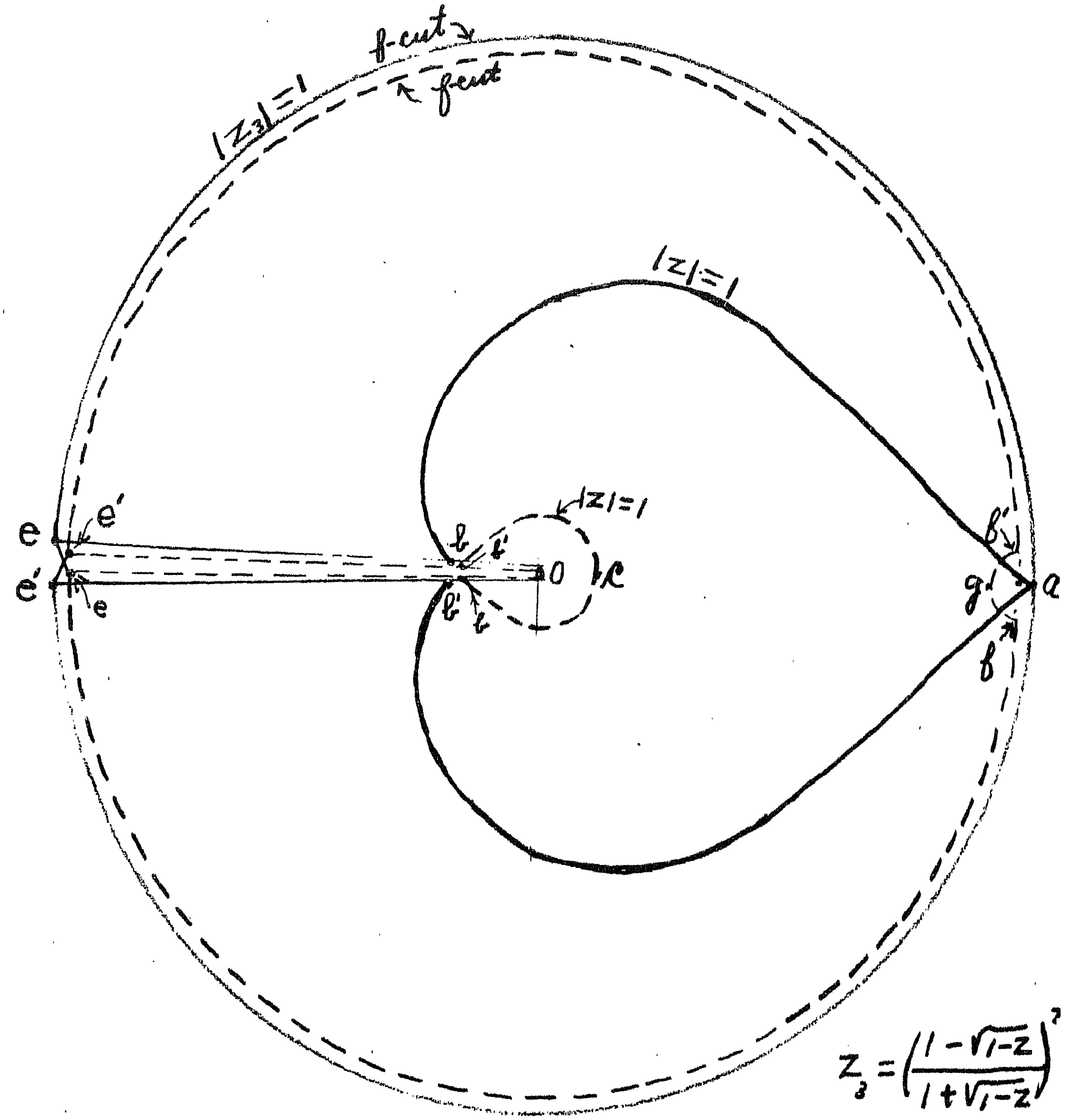
$$15)_b \quad F(\alpha, \alpha-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}; z_3) = (1+\sqrt{z_3})^{-2\alpha} F\left(\alpha, \beta, 2\beta; \frac{4\sqrt{z_3}}{(1+\sqrt{z_3})^2}\right).$$

where

$$15)_c \quad z_3 = \omega'^2 = \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2 \quad \text{so that} \quad z = \frac{4\omega'}{(1+\omega')^2} = \frac{4\sqrt{z_3}}{(1+\sqrt{z_3})^2}$$

This equation represents the entire z-plane having an f-cut (for the first member of $(15)_a$), upon a two sheeted Riemann's surface, each being a unit circle, with center O in common. Comparing Fig 5 with Fig 1 indicates the conformance. Both circles are cut along the radius Oe and connected as shown in fig 5.

The interior of the circle $|z-1|=1$ of the z-plane is represented upon the interior of the circle $|z_3|=1$ (upper sheet) heavy lines. The dotted lines are



See Fig 1

sheeted

m -plane

On upper sheet $z_3=\rho_3 e^{i\phi_3}$ where $-\pi<\phi_3<\pi$ $\quad 0<\rho_3<1$

On lower sheet $z_3=\rho_3 e^{i\phi_3}$ where $\pi<\phi_3<3\pi$ $\quad 0<\rho_3<1$

As ϕ_3 increases through π the point descends from upper to lower sheet; it returns to upper when ϕ_3 increases through 3π.

on the lower circular sheet whose interior repres all the region of the z-plane outside the circle $|z-1|=1$. When the point representing z in the z-plane moves across the arc of the circle $|z-1|=1$ the representative point in the z_3 plane moves from one sheet to the other, these being connected along the radius e b o b' e' which corresponds to the entire circle $|z-1|=1$. The infinite circle of z-, f g f' is an infinitesimal circle f g f' at a on the sheet. The circle $|z|=1$ of the z-plane a b c b' a lies on both sheets as indicated in fig 5. The origin O being on a cut is common to both sheets as is the line e b o and the line e' b' o.

$(15)_a$ is valid for all values of z in the z-plane with a cut, if attention is paid to the sheet upon which the point lies which represents $z_3 = \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2$.

Equ $(15)_e$ as a function of z_3 is valid, if z_3 is on the first sheet and inside the heart-shaped heavy curve (Fig 5) a f b' a representing $|z|=1$; but if z_3 is on the lower sheet, equ $(15)_e$ is then valid when z_3 lies inside the small dotted curve b c b'.

If in eq (9)$_a$ we place $\alpha = \frac{\alpha'}{2}$ and $\beta = \beta' - \frac{\alpha'}{2}$ and $\omega' = z_3$ it becomes after dropping the primes

$$16)_a \quad F(\alpha, \alpha-\beta+\tfrac{1}{2}, \beta+\tfrac{1}{2}; z_3) = (1+z_3)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\tfrac{1}{2}; \frac{4z_3}{(1+z_3)^2}\right)$$

Comparing this with (15)$_b$ gives

$$16)_b \quad (1+\sqrt{z_3})^{-2\alpha} F\left(\alpha, \beta, 2\beta; \frac{4\sqrt{z_3}}{(1+\sqrt{z_3})^2}\right) = (1+z_3)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\tfrac{1}{2}; \frac{4z_3}{(1+z_3)^2}\right)$$

On placing

$z = \dfrac{4\sqrt{z_3}}{(1+\sqrt{z_3})^2}$ so that $\dfrac{4z_3}{(1+z_3)^2} = \left(\dfrac{z}{2-z}\right)^2$ and $\dfrac{(1+\sqrt{z_3})^2}{1+z_3} = \dfrac{2}{2-z}$

this becomes

$$16)_c \quad F(\alpha, \beta, 2\beta, z) = \left(\frac{2}{2-z}\right)^{\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\tfrac{1}{2}; \left(\frac{z}{2-z}\right)^2\right)$$

To invert this first place $z = t$, then let $z = \left(\frac{t}{2-t}\right)^2$ so that $t = \dfrac{2\sqrt{z}}{1+\sqrt{z}}$. This gives on replacing α by 2α and $\gamma = \beta + \frac{1}{2}$

$$16)_d \quad F(\alpha, \alpha+\tfrac{1}{2}, \gamma; z) = (1+\sqrt{z})^{-2\alpha} F\left(2\alpha, \gamma-\tfrac{1}{2}, 2\gamma-1; \frac{2\sqrt{z}}{1+\sqrt{z}}\right)$$

Recapitulation of Non-linear transformations.

$17)_a$ $$F(\alpha, \beta, 2\beta; z) = \left(\frac{2-z}{2}\right)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, \beta+\frac{1}{2}; \left(\frac{z}{2-z}\right)^2\right)$$

$17)_b$ $$F(\alpha, \beta, 2\beta; z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(\alpha, \alpha-\beta+\frac{1}{2}, \beta+\frac{1}{2}; \left(\frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)^2\right)$$

$18)_a$ $$F(\alpha, \alpha+\tfrac{1}{2}, \gamma; z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, 2\alpha+1-\gamma, \gamma; \frac{1-\sqrt{1-z}}{1+\sqrt{1-z}}\right)$$

$18)_b$ $$F(\alpha, \alpha+\tfrac{1}{2}, \gamma; z) = (1+\sqrt{z})^{-2\alpha} F\left(2\alpha, \gamma-\frac{1}{2}, 2\gamma-1; \frac{2\sqrt{z}}{1+\sqrt{z}}\right)$$

$19)_a$ $$F(\alpha, \beta, 1+\alpha-\beta; z) = (1+\sqrt{z})^{-2\alpha} F\left(\alpha, \alpha-\beta+\frac{1}{2}, 2\alpha-2\beta+1; \frac{4\sqrt{z}}{(1+\sqrt{z})^2}\right)$$

$19)_b$ $$F(\alpha, \beta, 1+\alpha-\beta; z) = (1+z)^{-\alpha} F\left(\frac{\alpha}{2}, \frac{\alpha+1}{2}, 1+\alpha-\beta; \frac{4z}{(1+z)^2}\right)$$

$20)$ $$F\left(\alpha, \beta, \frac{\alpha+\beta+1}{2}; z\right) = F\left(\frac{\alpha}{2}, \frac{\beta}{2}, \frac{\alpha+\beta+1}{2}; 4z(1-z)\right)$$

$21)_a$ $$F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}; z) = F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{1-z}}{2}\right)$$

$21)_b$ $$F(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}; z) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-2\alpha} F\left(2\alpha, \alpha-\beta+\frac{1}{2}, \alpha+\beta+\frac{1}{2}; \frac{\sqrt{1-z}-1}{\sqrt{1-z}+1}\right)$$

$22)$ $$F\left(2\alpha, 2\beta, \alpha+\beta+\frac{1}{2}; \frac{1-\sqrt{z}}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} F(\alpha, \beta, \tfrac{1}{2}, z)$$
$$-2\sqrt{z}\cdot\frac{\sqrt{\pi}\,\Gamma(\alpha+\beta+\frac{1}{2})}{\Gamma(\alpha)\Gamma(\beta)} F\left(\alpha+\frac{1}{2}, \beta+\frac{1}{2}, \frac{3}{2}; z\right)$$

Certain finite series may be summed by these formulas, for example on equating coefficients of like powers of z in the expansion
$F(\gamma-\alpha,\gamma-\beta,\gamma;z)=(1-z)^{\alpha+\beta-\gamma}F(\alpha,\beta,\gamma;z)$ one obtains

$$23)\quad \sum_{t=0}^{K}\frac{\Gamma(t+\alpha)}{\Gamma(t+1)\Gamma(t+\gamma)\Gamma(1-\beta-t)\Gamma(t-K+1+\alpha+\beta-\gamma)\Gamma(1+K-t)}=$$

$$=\frac{\Gamma(\alpha)\quad\Gamma(1+\beta-\gamma)\Gamma(K+\gamma-\alpha)}{\Gamma(K+1)\Gamma(K+\gamma)\Gamma(\gamma-\alpha)\Gamma(1+\alpha+\beta-\gamma)\Gamma(1+\beta-\gamma-K)\Gamma(1-\beta)}$$

Similarly from $(21)_a$, one gets

$$24)\quad \sum_{t=0}^{K}\frac{2^{2t}\Gamma(t+\alpha)}{\Gamma(2t+1-K)\Gamma(1+K-t)\Gamma(t+\alpha-\beta+\frac{1}{2})\Gamma(1+\beta-t)}=$$

$$=\frac{2^{2\beta-2\alpha+1}\Gamma(\beta+\frac{1}{2})\Gamma(K+2\alpha)}{\Gamma(\alpha+\frac{1}{2})\Gamma(K+1)\Gamma(K+\alpha-\beta+\frac{1}{2})\Gamma(1+2\beta-K)}$$

The three special cases of (7) in which that series reduces to a hypergeometric series, give a single term for the s-series in (7), that is, the series (6). These cases are $\gamma=\alpha+\beta+\frac{1}{2}$, $\beta=\alpha+\frac{1}{2}$, $\gamma=2\beta$ which give eq (8), (19), (15). The series which may be summed in this manner are too numerous to tabulate.

Another case comes from the fact that if y_1 and y_2 are two independant solutions of

25) $$\begin{cases} \dfrac{d^2y}{dz^2} + P\dfrac{dy}{dz} + Qy = 0, \text{ the general solution of} \\ y''' + 3Py'' + [2P^2 + P' + 4Q]y' + [4PQ + 2Q']y = 0 \\ \text{is } y = C_1 y_1^2 + C_2 y_2^2 + C_3 y_1 y_2 \end{cases}$$

In this way it is found that

25)a $$\left[f(\alpha, \beta, \alpha+\beta+\tfrac{1}{2}, z)\right]^2 = 2\sqrt{\pi}\,\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})}\sum_{K=0}^{\infty} \frac{\Gamma(K+2\alpha)\Gamma(K+2\beta)\Gamma(K+\alpha+\beta)}{\Gamma(K+1)\Gamma(K+\alpha+\beta+\frac{1}{2})\Gamma(K+2\alpha+2\beta)}$$

Squaring the left side and equating coefficients gives

25)b $$\sum_{t=0}^{K} \frac{\Gamma(t+\alpha)\Gamma(t+\beta)\Gamma(K+\alpha-t)\Gamma(K+\beta-t)}{\Gamma(t+1)\Gamma(t+\alpha+\beta+\frac{1}{2})\Gamma(1+K-t)\Gamma(K+\alpha+\beta+\frac{1}{2}-t)} =$$

$$= \frac{2\sqrt{\pi}\,\Gamma(\alpha)\Gamma(\beta)\Gamma(K+2\alpha)\Gamma(K+2\beta)\Gamma(K+\alpha+\beta)}{\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})\Gamma(K+1)\Gamma(K+\alpha+\beta+\frac{1}{2})\Gamma(K+2\alpha+2\beta)}$$

Also placing $z = 1$ in (25) a gives

25)k $$\sum_{t=0}^{\infty} \frac{\Gamma(t+2\alpha)\Gamma(t+2\beta)\Gamma(t+\alpha+\beta)}{\Gamma(t+1)\Gamma(t+\alpha+\beta+\frac{1}{2})\Gamma(t+2\alpha+2\beta)} = \frac{\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{2\,\Gamma(\alpha+\frac{1}{2})\,\Gamma(\beta+\frac{1}{2})}$$

IV Integral Representations

Barnes' Contour integral.

1) $$f(\alpha,\beta,\gamma;z) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(\nu+\alpha)\Gamma(\nu+\beta)\Gamma(-\nu)}{\Gamma(\nu+\gamma)}(-z)^{\nu}\,d\nu \quad \text{where} \quad |\arg z|<\pi$$

where the path crosses the real axis by an infinitesimal detour to the left of the origin

Also if $R(\gamma) > R(\beta) > 0$ and $|z|<1$

2) $$\frac{\Gamma(\gamma-\beta)\Gamma(\beta)}{\Gamma(\gamma)} F(\alpha,\beta,\gamma;z) = \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-tz)^{-\alpha}\,dt$$

$$= 2\int_0^{\pi/2} \frac{\sin\theta^{2\gamma-2\beta-1}\cos\theta^{2\beta-1}}{(1-z\cos^2\theta)^{\alpha}}\,d\theta = 2\int_0^{\pi/2}\frac{\sin\theta^{2\beta-1}\cos\theta^{2\gamma-2\beta-1}}{(1-z\sin^2\theta)^{\alpha}}\,d\theta$$

Special cases of this:

2)$_a$ $$\int_0^{\pi}(1-\cos\theta)^{\gamma-2\beta}\cdot\sin\theta^{2\beta-1}(a_1^2-2a_1a_2\cos\theta+a_2^2)^{-\alpha}d\theta = \frac{2^{\gamma-1}\Gamma(\gamma-\beta)\Gamma(\beta)}{(a_1+a_2)^{2\alpha}\Gamma(\gamma)}F\left(\alpha,\beta,\gamma;\frac{4a_1a_2}{(a_1+a_2)^2}\right)$$

2)$_b$ $$a_1^{2\alpha}\int_0^{\pi}\sin\theta^{2\beta-1}(a_1^2-2a_1a_2\cos\theta+a_2^2)^{-\alpha}d\theta = \frac{\sqrt{\pi}\,\Gamma(\beta)}{\Gamma(\beta+\frac{1}{2})}F\left(\alpha,\alpha-\beta+\tfrac{1}{2},\beta+\tfrac{1}{2};\frac{a_2^2}{a_1^2}\right)$$

if $\left|\frac{a_2}{a_1}\right|<1$, $R(\beta)>0$

2)$_c$ $$a_1^{2\alpha}\int_0^{\pi}(1-\cos\theta)^{2\alpha-2\beta}\cdot\sin\theta^{2\beta-1}(a_1^2-2a_1a_2\cos\theta+a_2^2)^{-\alpha}d\theta =$$

$$= \frac{\sqrt{\pi}\,\Gamma(\beta)\Gamma(2\alpha-\beta)}{\Gamma(\alpha)\Gamma(\alpha+\frac{1}{2})}\left(\frac{a_1+a_2}{a_1}\right)^{2(\beta-\alpha)}F\left(\beta,\beta-\alpha+\tfrac{1}{2},\alpha+\tfrac{1}{2};\frac{a_2^2}{a_1^2}\right) \text{ if } \begin{pmatrix}|a_2|<|a_1| \\ R(2\alpha-\beta)>0\end{pmatrix}$$

If $x = R\cos\theta$, $r = R\sin\theta$, $R^2 = x^2 + r^2$

$$3)\quad \int_0^\infty t^\mu e^{-|x|t} J_\nu(rt)\,dt = \frac{\Gamma(\mu-\nu+1)\,P_\mu^\nu(\cos\theta)}{R^{\mu+1}} =$$

$$= \frac{r^\nu\,\Gamma(\mu+\nu+1)}{2^\nu (x^2+r^2)^{\frac{\mu+\nu+1}{2}}\,\Gamma(\nu+1)} F\left(\frac{\mu+\nu+1}{2}, \frac{\nu-\mu}{2}, \nu+1; \frac{r^2}{x^2+r^2}\right)$$

$$4)\quad \int_0^\infty t^\mu H_\rho^{(1)}(r_1 t)\,J_\nu(rt)\,dt =$$

$$= e^{i\frac{\pi}{2}(\mu+\nu-\rho)}\,\frac{1}{\pi}\,\frac{2^\mu r^\nu}{r_1^{\mu+\nu+1}} f\left(\frac{\mu+\nu+\rho+1}{2}, \frac{\mu+\nu-\rho+1}{2}, \nu+1; \frac{r^2}{r_1^2}\right)$$

if $r < r_1$

$H_\rho^{(1)}$ is Hankel's function of the first kind.

V A few relations of contiguity

Differentiating eq(16) I and substituting in (11) I gives

1) $z(1-z)\, f(\alpha+2,\beta+2,\gamma+2;z) + [\gamma-(\alpha+\beta+1)z]\, f(\alpha+1,\beta+1,\gamma+1;z)$

$$= \alpha\,\beta\, f(\alpha,\beta,\gamma;z)$$

The same equation is satisfied by $g(\alpha,\beta,\gamma;z)$

The general solution of the difference equation

2) $\begin{cases} z(1-z)\,u_{t+2} + [(1-2z)t+\gamma-(\alpha+\beta+1)z]\,u_{t+1} - (t+\alpha)(t+\beta)\,u_t = 0 \\ \text{is therefore} \\ u_t = C_1 f(\alpha+t,\beta+t,\gamma+t;z) + C_2\, g(\alpha+t,\beta+t,\gamma+t;z) \end{cases}$

where these functions represent the analytic continuation of f and g in case $|z|>1$.

Replacing f in terms of F, eq(1) becomes

3) $z(1-z)(\alpha+1)(\beta+1)\, F(\alpha+2,\beta+2,\gamma+2;z) +$

$$+ (\gamma+1)[\gamma-(\alpha+\beta+1)z]\, F(\alpha+1,\beta+1,\gamma+1;z)$$

$$= \gamma(\gamma+1)\, F(\alpha,\beta,\gamma;z)$$

The following are easilly derived from the definition (1) I

With α and β constant, γ varied.

4) $$\gamma[\gamma-1-(2\gamma-\alpha-\beta)z]\,F(\alpha,\beta,\gamma;z)+(\gamma-\alpha)(\gamma-\beta)z\,F(\alpha,\beta,\gamma+1;z)=$$
$$=\gamma(\gamma-1)(1-z)\,F(\alpha,\beta,\gamma-1;z).$$

With β and γ constant

5) $$[\gamma-\alpha-\beta+(\beta-\alpha)(1-z)]\,F(\alpha,\beta,\gamma;z)+\alpha(1-z)\,F(\alpha+1,\beta,\gamma;z)=$$
$$=(\gamma-\alpha)\,F(\alpha-1,\beta,\gamma;z).$$

With γ constant, α and β both varied

6) $$\ell\left\{\ell^2-1+[\ell(\alpha+\beta-1)+\alpha\beta+(\alpha-1)(\beta-1)](1-z)\right\}F(\alpha,\beta,\gamma;z)=$$

$$=(\ell+1)\alpha\beta(1-z)^2\,F(\alpha+1,\beta+1,\gamma;z)+(\ell-1)(\gamma-\alpha)(\gamma-\beta)\,F(\alpha-1,\beta-1,\gamma;z).$$

where $\ell = \gamma-\alpha-\beta$

7) $$(\alpha-\beta)\left\{\gamma(\alpha+\beta-1)+1-\alpha^2-\beta^2+[(\alpha-\beta)^2-1](1-z)\right\}F(\alpha,\beta,\gamma;z)=$$

$$=(\gamma-\alpha)(\alpha-\beta+1)\beta\,F(\alpha-1,\beta+1,\gamma;z)+(\gamma-\beta)(\alpha-\beta-1)\alpha\,F(\alpha+1,\beta-1,\gamma;z).$$

8) $$(\alpha-\beta)\,F(\alpha,\beta,\gamma;z)=\alpha\,F(\alpha+1,\beta,\gamma;z)-\beta\,F(\alpha,\beta+1,\gamma;z)$$

9) $$(\alpha-\beta)(1-z)\,F(\alpha,\beta,\gamma;z)=(\gamma-\beta)\,F(\alpha,\beta-1,\gamma;z)-(\gamma-\alpha)\,F(\alpha-1,\beta,\gamma;z)$$

10) $$(\gamma-\beta-1)\,F(\alpha,\beta,\gamma;z=(\gamma-\alpha-\beta-1)\,F(\alpha,\beta+1,\gamma;z)+\alpha(1-z)\,F(\alpha+1,\beta+1,\gamma;z)$$
$$=(\alpha-\beta-1)(1-z)\,F(\alpha,\beta+1,\gamma;z)+(\gamma-\alpha)\,F(\alpha-1,\beta+1,\gamma;z)$$

11) $$(\gamma-\alpha-\beta)\,F(\alpha,\beta,\gamma;z)=(\gamma-\alpha)\,F(\alpha-1,\beta,\gamma;z)-\beta(1-z)\,F(\alpha,\beta+1,\gamma;z)$$

12) $\alpha F(\alpha+1, \beta, \gamma; z) - (\gamma-1) F(\alpha, \beta, \gamma-1; z) = (\alpha+1-\gamma) F(\alpha, \beta, \gamma; z)$

13) $(1-z) F(\alpha, \beta, \gamma; z) - F(\alpha-1, \beta-1, \gamma; z) = \frac{(\alpha+\beta-\gamma-1)}{\gamma} z F(\alpha, \beta, \gamma+1; z)$

14) $\frac{(1-\beta)}{\gamma} z F(\alpha, \beta, \gamma+1; z) = F(\alpha-1, \beta-1, \gamma; z) - F(\alpha, \beta-1, \gamma; z)$

15) $(1-z) F(\alpha, \beta, \gamma; z) = F(\alpha, \beta-1, \gamma; z) + \frac{(\alpha-\gamma)}{\gamma} z F(\alpha, \beta, \gamma+1; z)$

There are 435 such relations between the 30 contiguous functions.

VI Associated Legendre Functions

1. Definitions and General Formulas

Every associated Legendre function of z, $P_\nu^\mu(z)$, $Q_\nu^\mu(z)$ or $T_\nu^\mu(z)$, $q_\nu^\mu(z)$ satisfies with its first derivative the two fundamental equations, with constant upper parameter, for all values of z

1) $(2\nu+1)zP_\nu^\mu(z) = (\nu+\mu)P_{\nu-1}^\mu(z) + (\nu-\mu+1)P_{\nu+1}^\mu(z)$

2) $(2\nu+1)(z^2-1)P_\nu'^\mu(z) = -(\nu+\mu)(\nu+1)P_{\nu-1}^\mu(z) + (\nu-\mu+1)\nu P_{\nu+1}^\mu(z).$

The following, derived from these are placed here for reference

2)$_a$ $(z^2-1)P_\nu'^\mu(z) = \nu z P_\nu^\mu(z) - (\nu+\mu)P_{\nu-1}^\mu(z) = -(\nu+1)zP_\nu^\mu(z) + (\nu-\mu+1)P_{\nu+1}^\mu(z)$

2)$_b$ $\left[\nu^2 + \frac{\mu^2}{z^2-1}\right]P_\nu^\mu = -(\nu+\mu)P_{\nu-1}'^\mu + \nu z P_\nu'^\mu$

2)$_c$ $\left[(\nu+1)^2 + \frac{\mu^2}{z^2-1}\right]P_\nu^\mu = (\nu-\mu+1)P_{\nu+1}'^\mu - (\nu+1)zP_\nu'^\mu$

2)$_d$ $(2\nu+1)\left[\nu(\nu+1) + \frac{\mu^2}{z^2-1}\right]P_\nu^\mu = -(\nu+\mu)(\nu+1)P_{\nu-1}'^\mu + (\nu-\mu+1)\nu P_{\nu+1}'^\mu$

From (1) and (2) one derives the differential equation

3) $\mathcal{D}_z\left[(1-z^2)y'(z)\right] + \left[\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right]y(z) = 0$

Legendre's function of the first kind, $P_\nu(z)$ is defined for $|z-1|<2$ by

4)a $$P_\nu(z) \equiv F\left(-\nu, \nu+1, 1; \frac{1-z}{2}\right) \equiv P_{-\nu-1}(z) \qquad |z-1|<2$$

The Legendre function of the second kind $Q_\nu(z)$ is single-valued at all points of the z-plane outside the circle $|z|=1$ cut from $-\infty$ to -1. Its definition is

4)b $$Q_\nu(z) \equiv \frac{1}{2z^{\nu+1}} f\left(\frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$$\equiv \frac{\sqrt{\pi}}{(2z)^{\nu+1}} \frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})} F\left(\frac{\nu}{2}+\frac{1}{2}, \frac{\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

For the analytic continuation of $Q_\nu(z)$ to the interior of the circle $|z|=1$ the cut is extended from -1 to 1 along the real axis. The continuation of $P_\nu(z)$ is made to points outside the circle $|z-1|=2$, the plane being cut from -1 to $-\infty$ along the real axis, except in the case where ν is any real integer, P_n being a polynomial in z, single valued in the entire plane.

Similarly when $\nu = n$ the cut for $Q_n(z)$ is only from -1 to 1.

The associated Legendre function $P_\nu^\mu(z)$ is here defined for unrestricted parameters μ and ν and for $|z-1|<2$ by

$$5)_a \quad P_\nu^\mu(z) \equiv \frac{(z^2-1)^{\frac{\mu}{2}}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\mu+1)\,\Gamma(\nu-\mu+1)}\,F\left(\mu-\nu,\mu+\nu+1,\mu+1;\frac{1-z}{2}\right)$$

or

$$5)_b \quad P_\nu^\mu(z) = \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)\,\Gamma(\nu-\mu+1)}\,F\left(-\nu,\nu+1,\mu+1;\frac{1-z}{2}\right)$$

The cut from $-\infty$ to -1 required for the continuation of these hg. functions also suffices for the factor $(z+1)^\mu$ where $-\pi<\arg(z+1)<\pi$ and $\arg(z+1)=0$ on the real axis to the right of the point -1. The principal value of $\arg z$ is between $-\pi$ and π being zero on the positive real axis so that when terms like z^ν or $\log z$ appear the cut must be extended from $-\infty$ to the origin. It must in fact be extended to $+1$ (along real axis) because of the factor $(z-1)^\mu$. The principal value of $\arg(z-1)$ is between $-\pi$ and π, being zero on the real axis to the right of $+1$.

The cut along the real axis from $-\infty$ to $+1$ renders $P_\nu^\mu(z)$ single valued in the plane thus

cut, and the function $Q_\nu^\mu(z)$ may now be so defined as to be single-valued in the plane cut in the same manner, that is wherever $P_\nu^\mu(z)$ is single-valued. This definition for all values of z is

6)a $$Q_\nu^\mu(z) \equiv -\frac{\pi}{2}\cot\mu\pi\left[P_\nu^\mu(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P_\nu^{-\mu}(z)\right]$$

The relation reciprocal to this is then found to be

6)b $$P_\nu^\mu(z) = \frac{\sin(\nu-\mu)\pi}{\pi\cos\nu\pi\cos\mu\pi}\left[Q_\nu^\mu(z) - Q_{-\nu-1}^\mu(z)\right]$$

The following relation derived from the differential eq (3) together with (5) and (6)a shows when P_ν^μ and Q_ν^μ are linearly independant solutions of (3)

7) $$(z^2-1)\left[Q_\nu^\mu(z)\,P_\nu^{\prime\mu}(z) - Q_\nu^{\prime\mu}(z)\,P_\nu^\mu(z)\right] = \cos\mu\pi\,\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}.$$

From the definition (5) it is found that the solution of (3), $P_{-\nu-1}^\mu(z)$ is not linearly independant of $P_\nu^\mu(z)$ for this definition gives

8)a $$P_{-\nu-1}^\mu(z) \equiv \frac{\sin(\nu+\mu)\pi}{\sin(\nu-\mu)\pi}P_\nu^\mu(z)$$

The particular solution of (3) $y = P_\nu^{-\mu}(z)$ is in general linearly independent of $P_\nu^{\mu}(z)$ the exception being when $\mu = m =$ any real integer. In that case (5) gives

$$8)_b \quad P_\nu^{-m}(z) = \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^{m}(z)$$

The definition (6)$_a$ shows that

$$9)_a \quad Q_\nu^{-\mu}(z) \equiv \frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)} Q_\nu^{\mu}(z)$$

The particular solution of (3) $y = Q_{-\nu-1}^{\mu}(z)$ is in general independent of $Q_\nu^{\mu}(z)$, the exception being when $\nu \pm \frac{1}{2} = n =$ any real integer. In that case (6)$_a$ gives

$$9)_b \quad Q_{-n-\frac{1}{2}}^{\mu}(z) = Q_{n-\frac{1}{2}}^{\mu}(z).$$

The two equations (6)$_a$ and (6)$_b$ are fundamental for the function pair $P_\nu^{\mu}(z)$ and $Q_\nu^{\mu}(z)$. Whenever an expansion or analytic continuation of one function has been obtained, the corresponding expansion of the other for the same range of z is given by the use of one or other of the equations (6)$_a$ or (6)$_b$. For this reason it is not always worth while to write

out both expansions.

The derivation of $(6)_b$ from $(6)_a$ is made by first replacing ν by $-\nu-1$ in $(6)_a$ and making use of the identity $(8)_a$. This gives by use of $(7)_b$ I

$$Q^{\mu}_{-\nu-1}(z) = -\frac{\pi}{2}\cot\mu\pi\left[\frac{\sin(\nu+\mu)\pi}{\sin(\nu-\mu)\pi}P^{\mu}_{\nu}(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P^{-\mu}_{\nu}(z)\right]$$

Subtracting this from $(6)_a$ gives $(6)_b$.

When $\mu \to m$ = an integer, eq. $(8)_b$ shows that the definition of $Q^m_\nu(z)$ in $(6)a$ becomes $\frac{0}{0}$ so that

$10)_a$ $$Q^{m}_{\nu}(z) = -\frac{1}{2}\mathcal{D}_\mu\left[P^{\mu}_{\nu}(z) - \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}P^{-\mu}_{\nu}(z)\right]_{\mu\to m}$$

Also $(9)_b$ shows that when $\nu = n-\frac{1}{2}$ where n is any integer, the expression $(6)_b$ for $P^{\mu}_{n-\frac{1}{2}}(z)$ becomes $\frac{0}{0}$ so that

$10)_b$ $$P^{\mu}_{n-\frac{1}{2}}(z) = -\frac{1}{\pi^2}\mathcal{D}_\nu\left[Q^{\mu}_{\nu}(z) - Q^{\mu}_{-\nu-1}(z)\right]_{\nu=n-\frac{1}{2}}$$

Much labor may be saved by noting that in practically all cases these limits have already been evaluated in section I. The brackets which vanish in $(10)_a$ and $(10)_b$ are seldom, if ever, anything except the difference of two f-functions of such

a nature that the bracket is a g-function except for some factor which in general does not vanish. The g-function is evaluated when its third parameter γ is an integer in eq (19) I.

When $z = x + i0$ the functions P_ν^μ and Q_ν^μ have different values from what they have at $z = x - i0$ when and only when $x < 1$. Since the real range $-1 < x < 1$ is important in the applications, it is appropriate to use Ferrer's notation $T_\nu^\mu(z)$.

The function $(z^2-1)^\nu$ (for non integral values of ν) or $(z+1)^\nu (z-1)^\nu$ is single-valued with the cut that has assumed for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ along the real axis from $-\infty$ to $+1$. But the function $(1-z^2)^\nu = (z+1)^\nu (1-z)^\nu$ is single-valued for a plane cut along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$, the principal value of $\arg(1-z)$ being zero on the real axis to the left of $+1$, and $-\pi$ just above the real axis to the right of $+1$, and $+\pi$ just below. Consequently as in section I eq (21)

$$\left.\begin{aligned} (z-1)^\nu &= (1-z)^\nu e^{\pm i\nu\pi} \\ \log(z-1) &= \log(1-z) \pm i\pi \end{aligned}\right\} \quad (11)$$

The upper or lower sign applies according as z is in the upper or lower half plane.

The function $T^{\mu}_{\nu}(z)$ differs from $P^{\mu}_{\nu}(z)$ by having the factor $(1-z^2)^{\frac{\mu}{2}}$ in place of the factor $(z^2-1)^{\frac{\mu}{2}}$ so that by (11), & (5) if z is any point in the plane

12) $$T^{\mu}_{\nu}(z) \equiv e^{\mp\frac{i\mu\pi}{2}} P^{\mu}_{\nu}(z)$$ The $\binom{\text{upper}}{\text{lower}}$ sign if z is in $\binom{\text{upper}}{\text{lower}}$ half plane

Hence if $|z-1|<2$

12)$_a$ $$T^{\mu}_{\nu}(z) = \frac{(1-z^2)^{\frac{\mu}{2}}\Gamma(\nu+\mu+1)}{2^{\mu}\Gamma(\mu+1)\Gamma(\nu-\mu+1)} F(\mu-\nu, \mu+\nu+1; \mu+1; \tfrac{1-z}{2})$$

$$= \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\mu+1)\Gamma(\nu-\mu+1)} F(-\nu, \nu+1, \mu+1, \tfrac{1-z}{2})$$

$T^{0}_{\nu}(z) = T_{\nu}(z) = P^{0}_{\nu}(z) = P_{\nu}(z) =$ Legendre's function of first kind.

The function $q^{\mu}_{\nu}(z)$ which is single-valued wherever $T^{\nu}_{\nu}(z)$ is (that is, in the z-plane cut from $-\infty$ to -1 and from $+1$ to $+\infty$) may be defined by the analogue of (6)$_a$

13)$_a$ $$q^{\mu}_{\nu}(z) \equiv -\frac{\pi}{2}\cot\mu\pi\left[T^{\mu}_{\nu}(z) - \cos\mu\pi \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} T^{-\mu}_{\nu}(z)\right]$$

so that

13)$_b$ $$T^{\mu}_{\nu}(z) = \frac{-2}{\pi\sin\mu\pi\cos\mu\pi}\left[q^{\mu}_{\nu}(z) - \cos\mu\pi\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} q^{-\mu}_{\nu}(z)\right]$$

The analogue of (6) b is

13)$_c$ $$T^{\mu}_{\nu}(z) = \frac{\sin(\nu-\mu)\pi}{\pi\cos\nu\pi\cos\mu\pi}\left[q^{\mu}_{\nu}(z) - q^{\mu}_{-\nu-1}(z)\right]$$

13)$_d$ $$q^{\mu}_{\nu}(z) = e^{\pm\frac{i\mu\pi}{2}}\cos\mu\pi\left[Q^{\mu}_{\nu}(z) \pm i\frac{\pi}{2}P^{\mu}_{\nu}(z)\right]$$ With the z-plane

cut for T and q, $\arg(1-z^2)$ is unchanged by reversing the sign of z so $T_\nu^\mu(-z)$ and $q_\nu^\mu(-z)$ are definite. When cut for P and Q, $P_\nu^\mu(ze^{-i\pi}) \neq P_\nu^\mu(ze^{i\pi})$ and ~~therefore~~ $P(-z)$ is ambiguous, also $Q(-z)$.

Corresponding to $(8)_a$ there is the same relation

$$14) \quad T_{-\nu-1}^{\mu}(z) = \frac{\sin(\nu+\mu)\pi}{\sin(\nu-\mu)\pi}\, T_\nu^\mu(z)$$

The analogue of $(8)_b$ is found in the two relations

$$15)_a \quad T_\nu^{-m}(z) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\, T_\nu^m(z)$$

$$15)_b \quad q_\nu^{-m}(z) = (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\, q_\nu^m(z)$$

m any real integer

The analogue of $(9)_b$ is found in the two relations where n is any real integer

$$16)_a \quad T_{-n-\frac{1}{2}}^{\mu}(z) = T_{n-\frac{1}{2}}^{\mu}(z)$$

$$16)_b \quad q_{-n-\frac{1}{2}}^{\mu}(z) = q_{n-\frac{1}{2}}^{\mu}(z)$$

There is no strict analogue of $(9)_a$ but this corresponds to $(13)_b$

The four functions P_ν^μ, Q_ν^μ, T_ν^μ and q_ν^μ all satisfy the fundamental equations (1) (2) and $(2)_a$ to $(2)_d$ and therefore the diff eq (3).

Since $T_\mu^0(z) = P_\mu(z)$ the two relations where m is a positive integer

$$17)_a \quad P_\nu^m(z) = (z^2-1)^{\frac{m}{2}} D_z^m P_\nu(z) \text{ and } Q_\nu^m(z) = (z^2-1)^{\frac{m}{2}} D_z^m Q_\nu(z),$$

give the similar relations

$$17)_b \quad \mathcal{T}_\nu^m(z) = (1-z^2)^{\frac{m}{2}} \mathcal{D}_z^m P_\nu(z) \quad \text{and} \quad q_\nu^m(z) = (1-z^2)^{\frac{m}{2}} \mathcal{D}_z^m q_\nu(z)$$

Also if $\nu = n$ = a positive integer

$$18) \quad P_n(z) = \frac{1}{2^n n!} \mathcal{D}_z^n (z^2-1)^n$$

When x is real and between -1 and 1

$$19) \quad q_\nu^\mu(x) = \tfrac{1}{2}\left[e^{-\frac{i\mu\pi}{2}} Q_\nu^\mu(x+i0) + e^{\frac{i\mu\pi}{2}} Q_\nu^\mu(x-i0)\right] \quad \text{by (12) and } (13)_d$$

The equation analogous to (7) which shows when $\mathcal{T}_\nu^\mu$ and q_ν^μ are linearly independant is

$$20) \quad (1-z^2)\left[q_\nu^\mu(z)\mathcal{T}_\nu^{\prime\mu}(z) - q_\nu^{\prime\mu}(z)\mathcal{T}_\nu^\mu(z)\right] = -\cos^2\mu\pi \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}$$

Eqs $17)_a$ and $(17)_b$ are special cases of the recurrence relations with constant lower parameter ν. These are derived by writting the definitions $(12)_a$ of $\mathcal{T}_\nu^\mu(z)$ and $(13)_a$ of $q_\nu^\mu(z)$ in terms of the f and g functions defined in section I, eqs (14) and (15).

Eqs $12)_a$ may be written

$$21)_a \quad \mathcal{T}_\nu^\mu(z) = \frac{(1-z^2)^{\mu/2} \sin(\mu-\nu)\pi}{\pi \; 2^\mu} f\left(\mu-\nu, \mu+\nu+1, \mu+1; \frac{1-z}{2}\right)$$

and eqn (13)$_a$ may be written

$$21)_b \quad q_\nu^\mu(z) = \frac{-(1-z^2)^{\frac{\mu}{2}}\cos\mu\pi}{2^{\mu+1}}\left\{\cos\nu\pi\, f(\mu-\nu, \mu+\nu+1, \mu+1; \tfrac{1-z}{2})\right.$$
$$\left. + \frac{\sin\nu\pi}{\pi}\, g(\mu-\nu, \mu+\nu+1, \mu+1; \tfrac{1-z}{2})\right\}$$

Since f and g satisfy the same relations (16)$_a$ and (16)$_b$ of I these two relations (21) show that T_ν^μ and q_ν^μ satisfy the same relations

$$22)_a \quad T_\nu^{\mu+m}(z) = (1-z^2)^{\frac{\mu+m}{2}} D_z^m\left[(1-z^2)^{-\frac{\mu}{2}} T_\nu^\mu(z)\right]$$

and

$$22)_b \quad q_\nu^{\mu+m}(z) = (1-z^2)^{\frac{\mu+m}{2}} D_z^m\left[(1-z^2)^{-\frac{\mu}{2}} q_\nu^\mu(z)\right]$$

which reduce to (17)$_b$ when $\mu=0$.
Multiplying (22)$_a$ by $e^{\pm i(\mu+m)\frac{\pi}{2}}$ gives by (11) and (12)

$$23)_a \quad P_\nu^{\mu+m}(z) = (z^2-1)^{\frac{\mu+m}{2}} D_z^m\left[(z^2-1)^{-\frac{\mu}{2}} P_\nu^\mu(z)\right]$$

whence by use of 6)$_a$

$$23)_b \quad Q_\nu^{\mu+m}(z) = (z^2-1)^{\frac{\mu+m}{2}} D_z^m\left[(z^2-1)^{-\frac{\mu}{2}} Q_\nu^\mu(z)\right]$$

From the relations (22) and (23) together with the differential equ (3) the following recurrence relations with constant lower parameter are derived.

24)a $$\frac{2\mu z}{\sqrt{1-z^2}} T_\nu^\mu(z) = (\nu+\mu)(\nu-\mu+1) T_\nu^{\mu-1}(z) + T_\nu^{\mu+1}(z)$$

24)b $$2\sqrt{1-z^2}\, T_\nu'^{\mu}(z) = -(\nu+\mu)(\nu-\mu+1) T_\nu^{\mu-1}(z) + T_\nu^{\mu+1}(z)$$

which may also be written

24)c $$\sqrt{1-z^2}\, T_\nu'^{\mu}(z) + \frac{\mu z}{\sqrt{1-z^2}} T_\nu^\mu(z) = T_\nu^{\mu+1}(z)$$

24)d $$\sqrt{1-z^2}\, T_\nu'^{\mu}(z) - \frac{\mu z}{\sqrt{1-z^2}} T_\nu^\mu(z) = -(\nu+\mu)(\nu-\mu+1) T_\nu^{\mu-1}(z)$$

These equations are also satisfied by $q_\nu^\mu(z)$

For $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ the relations are

25)a $$\frac{2\mu z}{\sqrt{z^2-1}} P_\nu^\mu(z) = (\nu+\mu)(\nu-\mu+1) P_\nu^{\mu-1}(z) - P_\nu^{\mu+1}(z)$$

25)b $$2\sqrt{z^2-1}\, P_\nu'^{\mu}(z) = (\nu+\mu)(\nu-\mu+1) P_\nu^{\mu-1}(z) + P_\nu^{\mu+1}(z)$$

25)c $$\sqrt{z^2-1}\, P_\mu'^{\nu}(z) - \frac{\mu z}{\sqrt{z^2-1}} P_\nu^\mu(z) = P_\nu^{\mu+1}(z)$$

25)d $$\sqrt{z^2-1}\, P_\mu'^{\nu}(z) + \frac{\mu z}{\sqrt{z^2-1}} P_\nu^\mu(z) = (\nu+\mu)(\nu-\mu+1) P_\nu^{\mu-1}(z)$$

Other Notations

If $z \neq \cos\theta$ $(0<\theta<\pi)$ the $P_\nu^m(z)$ and $Q_\nu^m(z)$ used and def. here are identical with the definitions of Hobson and Barnes which are used by Whittaker and Watson, where m is any real integer. For unrestricted μ and $z \neq x = \cos\theta$

26)$_a$ $$\left[P_\nu^\mu(z)\right]_{\substack{\text{Hobson}\\ \text{and Barnes}}} = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} P_\nu^{-\mu}(z)$$

26)$_b$ $$\left[Q_\nu^\mu(z)\right]_B = \frac{\sin(\nu+\mu)\pi}{\sin\nu\pi\cos\mu\pi} Q_\nu^\mu(z)$$

26)$_c$ $$\left[Q_\nu^\mu(z)\right]_H = \frac{e^{i\mu\pi}}{\cos\mu\pi} Q_\nu^\mu(z)$$

When $z = x = \cos\theta$ $\quad 0<\theta<\pi$

26)$_d$ $$\left[P_\nu^\mu(x)\right]_H = e^{\frac{i\mu\pi}{2}}\left[P_\nu^\mu(x+i0)\right]_H = \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)} T_\nu^{-\mu}(x)$$

so that

26)$_e$ $$\left[P_\nu^m(x)\right]_H = (-1)^m T_\nu^m(x)$$

26)$_f$ $$\left[Q_\nu^\mu(x)\right]_H = \frac{e^{-i\mu\pi}}{2}\left[e^{-\frac{i\mu\pi}{2}} Q_\nu^\mu(x+i0) + e^{\frac{i\mu\pi}{2}} Q_\nu^\mu(x-i0)\right]_H$$

$$= \frac{q_\nu^\mu(x)}{\cos\mu\pi} \quad \text{by (13)}_b$$

Hobson; Spherical Harmonics (pp 51, 52, 90–94, 195, 229).

The Kugel-function $K^{\nu,s}_{(x)}$ is defined by

$$26)_g\quad \begin{cases} K^{\nu s}_{(x)} = \dfrac{\Gamma(s+2\nu)}{\Gamma(2\nu)\Gamma(s+1)} F\left(-s, s+2\nu, \nu+\tfrac{1}{2}; \tfrac{1-x}{2}\right) \quad \text{so that} \\ K^{\nu,s}_{(x)} = \dfrac{\sqrt{2\pi}\,(1-x^2)^{-\frac{1}{2}(\nu-\frac{1}{2})}}{2^{\nu}\Gamma(\nu)} T^{\nu-\frac{1}{2}}_{\nu-\frac{1}{2}+s}(x) \text{ and } T^{\nu}_{\nu+s}(x) = \dfrac{2^{\nu}\Gamma(\nu+\frac{1}{2})(1-x^2)^{\frac{\nu}{2}}}{\sqrt{\pi}} K^{\nu+\frac{1}{2},s}_{(x)} \end{cases}$$

Particular solutions of (3) are $P^{\mu}_{\nu}(z)$, $Q^{\mu}_{\nu}(z)$, $P^{-\mu}_{\nu}(z)$ or $Q^{\mu}_{-\nu-1}(z)$. When $z = x = \cos\theta$ $\quad 0<\theta<\pi$, the solution may be taken as $T^{\mu}_{\nu}(x)$ $T^{-\mu}_{\nu}(x)$, $q^{\mu}_{\nu}(x)$, $q^{\mu}_{-\nu-1}(x)$. For some parameters a convenient pair of solutions is $T^{\mu}_{\nu}(x) = T^{\mu}_{\nu}(\cos\theta)$ and $T^{\mu}_{\nu}(-x) = T^{\mu}_{\nu}(\cos(\pi-\theta))$ where $0<\theta<\pi$. The following relations shows when these are linearly independent

$$26_h\quad \cos\mu\pi\, T^{\mu}_{\nu}(-z) = \cos\nu\pi\, T^{\mu}_{\nu}(z) + 2\frac{\sin(\mu-\nu)\pi}{\pi\cos\mu\pi} q^{\mu}_{\nu}(z) \quad \text{so that by (20)}$$

$$26)_i\quad (1-z^2)\left[T^{\mu}_{\nu}(z)\, D_z T^{\mu}_{\nu}(-z) - T^{\mu}_{\nu}(-z)\, D_z T^{\mu}_{\nu}(z)\right] = \frac{2\sin(\nu-\mu)\pi}{\sin(\nu+\mu)\pi\,\Gamma(-\nu-\mu)\Gamma(\nu-\mu+1)}$$

The "spindle" function, or "cone" function,

$$26)_j\quad T^{m}_{-\frac{1}{2}+i\nu}(x) = \frac{(-1)^m\cosh\nu\pi}{\pi}\left(\frac{1-x}{1+x}\right)^{\frac{m}{2}} \frac{\Gamma(m+\frac{1}{2}+i\nu)\Gamma(m+\frac{1}{2}-i\nu)}{\Gamma(m+1)} F\left(\tfrac{1}{2}+i\nu, \tfrac{1}{2}-i\nu, m+1; \tfrac{1-x}{2}\right)$$

is an even function of ν which is real when ν is real. In this case $T^{m}_{-\frac{1}{2}+i\nu}(x)$ and $T^{m}_{-\frac{1}{2}+i\nu}(-x)$ are independent and both real if ν is real.

2. The functions T_ν^μ, q_ν^μ and $K^{\nu s}$.

If for brevity we let

$$27)_a \quad A_\nu^\mu(z) \equiv \frac{2^\mu \Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu}{2}+1\right)} F\left(\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, \frac{1}{2}; z^2\right)$$

and

$$27)_b \quad B_\nu^\mu(z) \equiv \frac{2^{\mu+1}\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\nu-\mu+1}{2}\right)} z . F\left(\frac{\mu-\nu+1}{2}, \frac{\mu+\nu}{2}+1, \frac{3}{2}; z^2\right)$$

then the result of applying $(13)_b$ of III after replacing z by z^2 is to transform $12)_a$ into

$$28)_a \quad T_\nu^\mu(z) = (1-z^2)^{\frac{\mu}{2}}\left[A_\nu^\mu(z)\cos(\nu-\mu)\frac{\pi}{2} + B_\nu^\mu(z)\sin(\nu-\mu)\frac{\pi}{2}\right]$$

From this by $(13)_a$ we find

$$28)_b \quad q_\nu^\mu(z) = (1-z^2)^{\frac{\mu}{2}}\frac{\pi}{2}\cos\mu\pi\left[-A_\nu^\mu(z)\sin(\nu+\mu)\frac{\pi}{2} + B_\nu^\mu(z)\cos(\nu+\mu)\frac{\pi}{2}\right]$$

These are valid for general values of μ and ν when $|z|<1$. Inside this circle $T_\nu^\mu(z)$ and $q_\nu^\mu(z)$ are single-valued. The corresponding formulas for P and Q are

$$29)_a \quad P_\nu^\mu(z) = (z^2-1)^{\frac{\mu}{2}}\left[A_\nu^\mu(z)\cos(\nu-\mu)\frac{\pi}{2} + B_\nu^\mu(z)\sin(\nu-\mu)\frac{\pi}{2}\right]$$

$$29)_b \quad Q_\nu^\mu(z) = (z^2-1)^{\frac{\mu}{2}}\frac{\pi}{2}\cos\mu\pi\left[A_\nu^\mu(z)e^{\mp i(\nu+\mu+1)\frac{\pi}{2}} + B_\nu^\mu(z)e^{\mp i(\nu+\mu)\frac{\pi}{2}}\right]$$

which are valid inside the same circle with a cut along its real diameter. (The upper sign in upper half-plane;

Applying Gauss's Transformation $(1)_a$ II to A and B gives, since A is an even function of z

$$30)_a \quad \sin\mu\pi\, A_\nu^\mu(z) = \frac{\sin(\nu+\mu)\frac{\pi}{2}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\, F\left(\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, \mu+1; 1-z^2\right)$$

$$-(1-z^2)^{-\mu}\frac{2^\mu \sin(\nu-\mu)\frac{\pi}{2}}{\Gamma(-\mu+1)}\, F\left(\frac{-\mu-\nu}{2}, \frac{\nu-\mu+1}{2}, -\mu+1; 1-z^2\right)$$

and since B is an odd function of z

$$30)_b \quad \sin\mu\pi\, B_\nu^\mu(z) = \frac{-\cos(\nu+\mu)\frac{\pi}{2}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\, F\left(\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, \mu+1; 1-z^2\right)$$

$$+(1-z^2)^{-\mu}\frac{2^\mu \cos(\nu-\mu)\frac{\pi}{2}}{\Gamma(-\mu+1)}\, F\left(\frac{-\mu-\nu}{2}, \frac{\nu-\mu+1}{2}, -\mu+1; 1-z^2\right)$$

For general values of μ and ν this is only valid when z is inside the right lobe of the lemniscate (Fig 1) whose equation is $|z^2-1|=1$. (see eq (50) & (58) below)

Hence if z is inside the loop of the lemniscate for which $R(z)>0$ the eq 28)$_a$ transforms into

$$31)_a \quad T_\nu^\mu(z) = \frac{(1-z^2)^{\frac{\mu}{2}}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\, F\left(\frac{\mu-\nu}{2}, \frac{\nu+\mu+1}{2}, \mu+1; 1-z^2\right)$$

(when z is inside the left lobe see $(52)_b$)

The continuation of $q_\nu^\mu(z)$ to the inside of this loop may be written

$$31)_b \quad q_{\mu+\sigma}^{\mu}(z) = -\frac{(1-z^2)^{\frac{\mu}{2}}\, 2^{\mu-1}\, \Gamma(\frac{\sigma}{2}+\mu+1)}{\pi\, \Gamma(\frac{\sigma+1}{2})} \Big\{ \sin\frac{\sigma\pi}{2}\, g(-\tfrac{\sigma}{2}, \mu+\tfrac{\sigma+1}{2}, \mu+1; 1-z^2)$$

$$+\pi \cos\mu\pi \cos(\mu+\tfrac{\sigma}{2})\pi\, (1-z^2)^{-\mu} f(-\tfrac{\sigma}{2}-\mu, \tfrac{\sigma+1}{2}, -\mu+1; 1-z^2) \Big\}$$

The special case of this when $\mu \to m = 0,1,2,3,\cdots$ is by $(18)_a$, (19) and $(8)_c$ of I

$$31)_b' \quad q_{m+\sigma}^{m}(z) = -\frac{(1-z^2)^{\frac{m}{2}}\, 2^{m-1}\, \Gamma(\frac{\sigma}{2}+m+1)}{\Gamma(\frac{\sigma+1}{2})} \Bigg\{ \frac{\Gamma(m+\frac{\sigma+1}{2})}{\Gamma(\frac{\sigma}{2}+1)\,\Gamma(m+1)} F(-\tfrac{\sigma}{2}, m+\tfrac{\sigma+1}{2}, m+1; 1-z^2) \log(1-z^2)$$

$$+\sum_{t=-1}^{t=-m<0} (1-z^2)^t \frac{\Gamma(-t)\,\Gamma(t+m+\frac{\sigma+1}{2})}{\Gamma(t+m+1)\,\Gamma(\frac{\sigma}{2}+1-t)}$$

$$+\sum_{t=0}^{\infty} \frac{(-1)^t (1-z^2)^t\, \Gamma(t+m+\frac{\sigma+1}{2})}{\Gamma(t+1)\,\Gamma(t+m+1)\,\Gamma(\frac{\sigma}{2}+1-t)} \Big[\psi(\tfrac{\sigma}{2}+1-t) + \psi(t+m+\tfrac{\sigma+1}{2}) - \psi(t+m+1) - \psi(t+1) \Big] \Bigg\}$$

When $\sigma = 2s$ where $s = 0,1,2,3,\cdots$ the terms of this infinite series for which $t > s$ become by $(8)_g$ I

$$(-1)^s \sum_{t=s+1}^{\infty} \frac{(1-z^2)^t\, \Gamma(t+m+s+\frac{1}{2})\, \Gamma(t-s)}{\Gamma(t+1)\, \Gamma(t+m+1)}$$

The general eq (31)$_\xi$ may be written

$$31)''_\xi \quad q^{\mu}_{\mu+\sigma}(z) = -(1-z^2)^{\frac{\mu}{2}} \frac{\pi}{2} \cot \mu\pi \left[\frac{\Gamma(2\mu+\sigma+1)}{2^{\mu}\Gamma(\sigma+1)\Gamma(\mu+1)} F(-\tfrac{\sigma}{2}, \mu+\tfrac{\sigma+1}{2}, \mu+1; 1-z^2) \right.$$

$$\left. - \frac{(1-z^2)^{-\mu} 2^{\mu} \cos\mu\pi}{\Gamma(-\mu+1)} F(-\mu-\tfrac{\sigma}{2}, \tfrac{\sigma+1}{2}, -\mu+1; 1-z^2) \right]$$

which is more convenient when σ only is an integer.

The case frequently occurring in applications is where $\nu = \mu + s$ where $s = 0, 1, 2, 3, \ldots$. In that case the two equations

$$32)_q \quad T^{\mu}_{\mu+s}(z) = \frac{(1-z^2)^{\frac{\mu}{2}}\Gamma(s+2\mu+1)}{2^{\mu}\Gamma(\mu+1)\Gamma(s+1)} F(-s, s+2\mu+1, \mu+1; \tfrac{1-z}{2})$$

and

$$32)_\xi \quad T^{\mu}_{\mu+s}(z) = \frac{(-1)^s(1-z^2)^{\frac{\mu}{2}}\Gamma(s+2\mu+1)}{2^{\mu}\Gamma(\mu+1)\Gamma(s+1)} F(-s, s+2\mu+1, \mu+1; \tfrac{1+z}{2}) = (-1)^s T^{\mu}_{\mu+s}(-z)$$

are valid in the entire z-plane, cut along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$. The F-functions are polynomials. These equations show that $T^{\mu}_{\mu+2s}(z)$ is an even function of z and $T^{\mu}_{\mu+2s+1}(z)$ an odd.

For the case $s=0$, $\mu = -\frac{1}{2}$, $T^{\mu}_{\mu+2s} = \infty$, but the factor $\Gamma(2\mu+1)$ may be removed. (See eq 39 below)

The following equations are also valid in the entire z-plane since the F-functions are polynomials.

33)$_a$ $$T^{\mu}_{\mu+2s}(z) = \frac{2^{\mu}(1-z^2)^{\frac{\mu}{2}}(-1)^s\,\Gamma(s+\mu+\frac{1}{2})}{\sqrt{\pi}\;\Gamma(s+1)}\,F(-s, s+\mu+\tfrac{1}{2}; \tfrac{1}{2}; z^2)$$

$$= \frac{2^{\mu}(1-z^2)^{\frac{\mu}{2}}\,\Gamma(s+\mu+1)\,\Gamma(s+\mu+\frac{1}{2})}{\Gamma(\mu+1)\;\Gamma(s+1)\;\Gamma(s+\frac{1}{2})}\,F(-s, s+\mu+\tfrac{1}{2}, \mu+1; 1-z^2)$$

33)$_b$ $$T^{\mu}_{\mu+2s+1}(z) = \frac{2^{\mu+1}(1-z^2)^{\frac{\mu}{2}}(-1)^s\,\Gamma(s+\mu+\frac{3}{2})}{\sqrt{\pi}\;\Gamma(s+1)}\,.\,z\,.\,F(-s, s+\mu+\tfrac{3}{2}, \tfrac{3}{2}; z^2)$$

$$= \frac{2^{\mu}(1-z^2)^{\frac{\mu}{2}}\,\Gamma(s+\mu+1)\,\Gamma(s+\mu+\frac{3}{2})}{\Gamma(\mu+1)\;\Gamma(s+1)\;\Gamma(s+\frac{3}{2})}\,.\,z\,.\,F(-s, s+\mu+\tfrac{3}{2}, \mu+1; 1-z^2)$$

The set of functions $T^{\mu}_{\mu+s}(x)$ when $s = 0, 1, 2, 3, \ldots$ constitute a complete set of normal functions (if $R(\mu) > -1$) for the ~~real~~ range $-1 < x < 1$ for,

34)$_a$ $$\int_{-1}^{1} T^{\mu}_{\mu+s}(x)\, T^{\mu}_{\mu+s'}(x)\,dx = 0 \quad \text{if } s' \neq s$$

and,

34)$_b$ $$\int_{-1}^{1} \left[T^{\mu}_{\mu+s}(x)\right]^2 dx = \frac{\Gamma(s+2\mu+1)}{(s+\mu+\frac{1}{2})\,\Gamma(s+1)}$$

The set of even functions $T^{\mu}_{\mu+2s}(x)$ constitute a complete

set of orthogonal functions for the range $0 < x < 1$. Another complete set for this range is the set $T^{\mu}_{\mu+2s+1}(x)$. If (32) be divided by $\Gamma(s+2\mu+1)$ the orthogonal properties remain in the case where $\mu = -\frac{1}{2}$. To prove (34)$_a$ and (34)$_b$ we find from the diff eq (3) that

35)$_a$ $$(\sigma-\sigma_0)(\sigma+\sigma_0+2\mu+1)\int_0^x T^{\mu}_{\mu+\sigma}(x) \cdot T^{\mu}_{\mu+\sigma_0}(x)\,dx = L(x) - L(0)$$

where

35)$_b$ $$L(x) \equiv (1-x^2)\left[T'^{\mu}_{\mu+\sigma_0}(x)\, T^{\mu}_{\mu+\sigma}(x) - T^{\mu}_{\mu+\sigma_0}(x)\, T'^{\mu}_{\mu+\sigma}(x)\right]$$

The integral in (35)$_a$ is taken along the positive real axis up to a point x less than 1. For this range (31)$_a$ applies, and by its use we find when $1-x$ is small that

35)$_c$ $$L(x) = \frac{(\sigma-\sigma_0)(\sigma+\sigma_0+2\mu+1)\,\Gamma(\sigma+2\mu+1)\,\Gamma(\sigma_0+2\mu+1)\,x(1-x^2)^{\mu+1}}{2^{2\mu+1}\,\Gamma(\mu+1)\,\Gamma(\mu+2)\,\Gamma(\sigma+1)\,\Gamma(\sigma_0+1)}\left[1 + \text{zero}(1-x^2)\right]$$

This vanishes when $x \to 1$ if $R(\mu) > -1$ if σ and σ_0 are real and not negative. (In exceptional case $\mu = -\frac{1}{2}$ see B^{os} and eq 39).
From (28)$_a$ it is found that

35)$_d$ $$L(0) = \frac{2^{2\mu+1}}{\pi}\left[\frac{\Gamma(\mu+\frac{\sigma_0+1}{2})}{\Gamma(\frac{\sigma_0}{2}+1)}\cos\sigma_0\frac{\pi}{2}\,\frac{\Gamma(\mu+\frac{\sigma}{2}+1)}{\Gamma(\frac{\sigma+1}{2})}\sin\sigma\frac{\pi}{2} - \frac{\Gamma(\mu+\frac{\sigma_0}{2}+1)}{\Gamma(\frac{\sigma_0+1}{2})}\sin\sigma_0\frac{\pi}{2}\cdot\frac{\Gamma(\mu+\frac{\sigma+1}{2})}{\Gamma(\frac{\sigma}{2}+1)}\cos\sigma\frac{\pi}{2}\right]$$

This shows that $L(\sigma)$ vanishes if σ and σ_0 are both even or both odd, non-negative integers. Giving σ_0 a fixed non-negative integral value and then dividing $(35)_a$ by $\sigma-\sigma_0$ one obtains when $\sigma\to\sigma_0$ the formula $(34)_b$ and also $(34)_a$.

The formal development of $f(x)$ is

36) $$\begin{cases} f(x) = \sum_{s=0}^{\infty} f_s \, T_{\mu+s}^{\mu}(x) \quad \text{for } -1<x<1 \qquad R(\mu)>-1 \\ \text{where} \\ f_s = \dfrac{(s+\mu+\frac{1}{2})}{\Gamma(s+2\mu+1)} s! \displaystyle\int_{-1}^{1} f(x)\, T_{\mu+s}^{\mu}(x)\, dx \end{cases}$$

A well-known case is (see eq $(67)_b$ below)

36)' $$\frac{(1-x^2)^{\frac{\mu}{2}}}{(1-2xh+h^2)^{\mu+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^{\mu}\Gamma(\mu+\frac{1}{2})} \sum_{s=0}^{\infty} h^s \, T_{\mu+s}^{\mu}(x) \quad \text{-- if } |h|<1$$

The development (36) is in fact a series of polynomials since every function $T_{\mu+s}^{\mu}(x)$ in the series is a polynomial multiplied by the factor $(1-x^2)^{\frac{\mu}{2}}$ which being independent of s may be divided out and absorbed in the first member.

This is equivalent to the development in orthogonal polynomials, - the Kugel functions eq (26).

The infinite set of polynomials $K^{\nu,s}_{(x)}$, $s=0,1,2,3,---$ are orthogonal for the range $-1<x<1$ with a weighting function $(1-x^2)^{\nu-\frac{1}{2}}$ so that

$$37)_a \quad \begin{cases} \int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}} K^{\nu,s}_{(x)} K^{\nu,s_1}_{(x)}\,dx = 0 \text{ if } s_1 \neq s \\ \int_{-1}^{1}(1-x^2)^{\nu-\frac{1}{2}} \left[K^{\nu,s}_{(x)}\right]^2 dx = \dfrac{2\pi\,\Gamma(s+2\nu)}{2^{2\nu}(s+\nu)\Gamma^2(\nu)\,\Gamma(s+1)} \end{cases}$$

The development formula is

$$37)_b \quad \begin{cases} f(x) = \sum_{s=0}^{\infty} f_s K^{\nu,s}_{(x)} \quad --- \quad -1<x<1 \\ \text{where} \\ f_s = \dfrac{2^{\nu-1}\Gamma^2(\nu)}{\pi}\dfrac{(s+\nu)\,s!}{\Gamma(s+2\nu)}\int_{-1}^{1} f(x)(1-x^2)^{\nu-\frac{1}{2}} K^{\nu,s}_{(x)}\,dx \end{cases}$$

The equ (36)' takes the form

$$37)_c \quad (1-2xh+h^2)^{-\nu} = \sum_{s=0}^{\infty} h^s K^{\nu,s}_{(x)} \qquad |h|<1$$

The even functions of x, $K^{\nu,2s}_{(x)}$, $s=0,1,2,3,---$ make a closed set of orthogonal polynomials for the range $0<x<1$ as also do the odd functions $K^{\nu,2s+1}_{(x)}$, $s=0,1,2,3,--$, the definitions

being

[illegible] $$K^{\nu,2s}_{(x)} = \frac{(-1)^s\,\Gamma(s+\nu)}{\Gamma(\nu)\,\Gamma(s+1)}\,F(-s,\ s+\nu,\ \tfrac{1}{2},\ x^2)$$

$$= \frac{\Gamma(2s+2\nu)}{\Gamma(2\nu)\,\Gamma(2s+1)}\,F(-s,\ s+\nu,\ \nu+\tfrac{1}{2};\ 1-x^2)$$

38) $$K^{\nu,2s+1}_{(x)} = \frac{(-1)^s\,\Gamma(s+\nu+1)}{\Gamma(\nu)\,\Gamma(s+1)} \cdot 2x\,F(-s,\ s+\nu+1,\ \tfrac{3}{2};\ x^2)$$

$$= \frac{\Gamma(2s+2\nu+1)}{\Gamma(2\nu)\,\Gamma(2s+2)} \cdot x \cdot F(-s,\ s+\nu+1;\ \nu+\tfrac{1}{2};\ 1-x^2)$$

Since $K^{\nu 0}_{(x)} = 1$ and $\lim\limits_{\nu\to 0} \Gamma(\nu)\,K^{\nu,s}(\cos\theta) = 2\,\dfrac{\cos s\theta}{s}$, $s>0$ the expansion (37) becomes a Fourier's cosine series.

39) $$f(\theta) = \frac{1}{\pi}\int_0^{\pi} f(\theta)\,d\theta + \sum_{s=1}^{\infty} \cos s\theta \cdot \frac{2}{\pi}\int_0^{\pi} f(\theta_1)\cos s\theta_1\,d\theta_1$$

This case is the one noted above in which $\mu = -\frac{1}{2}$ for which the function $T^{\mu}_{(x)}$ $\mu+0$ is meaningless.

In applications to spherical and spheroidal harmonics μ is an integer, and the development (36) is usually written in the form

40) $$\begin{cases} f(x) = \displaystyle\sum_{n=m}^{\infty} f_n\, T^{m}_{n}(x) & \text{for } -1 < x < 1 \\ \text{where} & \\ f_n = (n+\tfrac{1}{2})\dfrac{(n-m)!}{(n+m)!}\displaystyle\int_{-1}^{1} f(x)\,T^{m}_{n}(x)\,dx & \end{cases}$$

Another case in which $T_\nu^\mu(z)$ reduces to a polynomial is where ν only is an integer. If $\nu = n = 0, 1, 2, 3, \cdots$ the following expressions are valid in the entire z-plane cut along the real axis from $-\infty$ to -1 and from $+1$ to $+\infty$

41) $$T_n^\mu(z) = \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} \frac{\Gamma(n+\mu+1)}{\Gamma(n-\mu+1)\Gamma(\mu+1)} F\left(-n, n+1, \mu+1; \frac{1-z}{2}\right)$$

$$= \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} \left(\frac{1+z}{2}\right)^{n} \frac{\Gamma(-n+\mu+1)}{\Gamma(n-\mu+1)\Gamma(\mu+1)} F\left(-n, \mu-n, \mu+1; \frac{z-1}{z+1}\right)$$

$$= \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} (-1)^n F\left(-n, n+1, -\mu+1; \frac{1+z}{2}\right)$$

$$= \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} \left(\frac{1-z}{2}\right)^{n} (-1)^n F\left(-n, -\mu-n, -\mu+1; \frac{z+1}{z-1}\right)$$

Replacing the factor $\left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}}$ by $\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}}$ gives the expressions for $P_n^\mu(z)$ valid in the entire z-plane with a cut from $-\infty$ to $+1$ along the real axis

Eq.(35)$_d$ shows that $T_n^\mu(x)$, $n = 0, 1, 2, 3, \cdots$ is not a set of orthogonal functions for $-1 < x < 1$ except in the case considered above where μ is an integer, and $n \geq m$. It will be shown in the next section that the only values of ν giving finite solutions of (3) at $z=1$ and $z=-1$ when $\mu = m$, are $\nu = n \geq m$ and the solutions are $T_n^m(z)$ or $P_n^m(z)$.

3. Homographic Transformations of $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ starting from argument $\frac{1-z}{2}$, into $\frac{2}{1-z}$, $\frac{1+z}{2}$, $\frac{2}{1+z}$, $\frac{z-1}{z+1}$, $\frac{z+1}{z-1}$.

In section II, the homographic transformations of the hypergeometric function $F(\alpha,\beta,\gamma;z)$ covered the z-plane with an f-cut in three different ways.

The definition $(5)_a$ of $P_\nu^\mu(z)$ together with the general definition $(6)_a$ of $Q_\nu^\mu(z)$ suffices for all points inside the circle of radius 2 with center at +1. (i.e for $|z-1|<2$) the plane being cut from $-\infty$ to $+1$. The remainder of the plane outside this circle $|z-1|>2$ will require expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ in terms of hg.-functions with argument $\frac{2}{z-1}$.

Expressions for P_ν^μ & Q_ν^μ will also be obtained in terms of hg-functions of $\frac{1+z}{2}$, valid when $|1+z|<2$, that is, inside a circle of diameter 2 with the center at -1. The remainder of the plane outside this circle requires hg-functions of $\frac{2}{1+z}$. A third way of covering the plane is with functions of $\frac{z-1}{z+1}$ valid when $R(z)>0$ and of $\frac{z+1}{z-1}$ valid when $R(z)<0$. The six subdivisions of the z-plane are shown in fig 1 below.

The required formulas may all be found as

special cases of section II, proper consideration being given to the fact that the variables are different and the cuts therefore different in that section from the cuts in the z-plane here. Since we here start with functions of $\frac{1-z}{2}$ instead of z it is perhaps safer to derive these five new cases by use of Euler's and Gauss's transformations (eq (2) and $(1)_a$ respectively of II) making use when desirable of the identity (3) I. The principal source of error in these is confusion as to the meaning of $(1-z)^{\nu}$ when z is replaced by its various homographic substitutions. The explanation of cuts and principal values on pages 43 and 47 of this section, together with eq. (11) should be sufficient to resolve any ambiguity of this kind.

The procedure may be sketched as follows:— Applying Gauss's transformation to the hg-function of $\frac{1-z}{2}$ in $(5)_a$ gives two hg functions of $\frac{1+z}{2}$. Then applying Euler's transformation to these functions of $\frac{1+z}{2}$ converts each into a hg-function of $\frac{z+1}{z-1}$. If however these two transformations of $(5)_a$ are made in different order, that of Euler's

converts $(5)_a$ into a single hg-function of $\frac{z-1}{z+1}$. Applying Gauss's formula to this gives two hg-functions of $\frac{2}{1+z}$. The sixth and remaining form is obtained by applying Euler's formula which converts each of these hg-functions of $\frac{2}{1+z}$ into an hg-function of $\frac{2}{1-z}$.

There are six formulas of this type for $P_\nu^\mu(z)$ including $(5)_a$, and by $(6)_a$ there are six for $Q_\nu^\mu(z)$. A few of these are tabulated here, all others being obtainable from these by use of $(6)_a$ or $(6)_b$.

42) $$Q_\nu^\mu(z) = \frac{2^\nu \cos\mu\pi (z^2-1)^{\frac{\mu}{2}}}{(z-1)^{\nu+\mu+1}} f\left(\nu+1, \nu+\mu+1, 2\nu+2; \frac{2}{1-z}\right) \quad \text{if } |z-1|>2$$

43) $$Q_\nu^\mu(z) = \frac{2^\nu \cos\mu\pi (z^2-1)^{\frac{\mu}{2}}}{(z+1)^{\nu+\mu+1}} f\left(\nu+1, \nu+\mu+1, 2\nu+2; \frac{2}{1+z}\right) \quad \text{if } |z+1|>2$$

44) $$P_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}} \sin\nu\pi \sin(\nu-\mu)\pi}{2^\mu \, \pi^2 \cos\mu\pi} g\left(\mu-\nu, \nu+\mu+1, \mu+1; \frac{1+z}{2}\right) \quad \text{if } |z+1|<2$$

45) $$P_\nu^\mu(z) = \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} \left(\frac{z+1}{2}\right)^\nu \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)} F\left(-\nu, \mu-\nu, \mu+1; \frac{z-1}{z+1}\right) \quad \text{if } R(z)>0$$

46) $$Q_\nu^\mu(z) = \frac{\pi \cos\mu\pi \cos\nu\pi}{\sin(\nu-\mu)\pi} P_\nu^\mu(z)$$

$$- \frac{\sin(\nu+\mu)\pi \, \Gamma(\nu+\mu+1)}{2\pi\, \Gamma(-\nu, \ldots} \left(\frac{z+1}{2}\right)^\nu \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} g\left(-\nu, \mu-\nu, \mu+1, \frac{z-1}{z+1}\right)$$

if $R(z)>0$

47) $Q_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}}}{(z-1)^{\mu+\nu+1}} \frac{2^\nu}{\Gamma(\nu+1)\Gamma(\nu-\mu+1)} g(\nu+1, \nu+\mu+1, \mu+1; \frac{z+1}{z-1})$ if $R(z) < 0$.

The special cases of (44) when $\nu = \mu + s$, $s = 0, 1, 2, \ldots$ are obtained from (32) and (33) where $(1-z^2)^{\frac{\mu}{2}}$ is replaced by $(z^2-1)^{\frac{\mu}{2}}$. The special cases where ν only is an integer are given by (41) where $\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}}$ replaces $\left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}}$.

In the case where the upper parameter only, is an integer eq (44) becomes for $\mu = m = 0, 1, 2, 3, \ldots$ by reference to (19) I.

44)' $$P_\nu^m(z) = \frac{(z^2-1)^{\frac{m}{2}}}{2^m}\left(\frac{\sin\nu\pi}{\pi}\right)^2 \left\{ \sum_{s=1}^{m>0} (-1)^s \left(\frac{2}{1+z}\right)^s \frac{\Gamma(s)\Gamma(m-\nu-s)\Gamma(m+\nu+1-s)}{\Gamma(m+1-s)} \right.$$

$$- f\left(m-\nu, m+\nu+1, m+1; \frac{1+z}{2}\right) \log\left(\frac{1+z}{2}\right)$$

$$- \sum_{s=0}^{\infty} \left(\frac{1+z}{2}\right)^s \frac{\Gamma(s+m-\nu)\Gamma(s+m+\nu+1)}{\Gamma(s+1)\Gamma(s+m+1)} \Big[\psi(s+m-\nu) + \psi(s+m+\nu+1)$$

$$\left. - \psi(s+m+1) - \psi(s+1) \Big] \right\}$$

where the first sum is absent if $m = 0$, and where $|z+1| < 2$

This shows that the only values of ν for which $P_\nu^m(z)$ and its derivatives are finite when $z \to -1$ are $\nu = n \geq m$. Also $Q_\nu^m(z)$ cannot be finite for $z = +1$ as shown in the following 3 equations. Hence the only values of ν when $\mu = m = 0, 1, 2, 3$ for which (3) possesses solutions finite at $z = 1$ and $z = -1$ are $\nu = n \geq m$ and the solution is $P_n^m(z)$ or $T_n^m(z)$

The case of (46) in which $\mu = m = 0, 1, 2, 3$ -- becomes by (19) I

46)' $$Q_\nu^m(z) = P_\nu^m(z)\left[\frac{1}{2}\log\left(\frac{z+1}{z-1}\right) + \pi\cot\nu\pi\right]$$

$$+\frac{1}{2}\Gamma(\nu+1)\Gamma(\nu+m+1)\left(\frac{z+1}{2}\right)^{\nu}\left(\frac{z-1}{z+1}\right)^{\frac{m}{2}}\left\{\sum_{s=1}^{m>0}\frac{(-1)^s\left(\frac{z+1}{z-1}\right)^s\Gamma(s)}{\Gamma(s+\nu+1)\Gamma(s+\nu-m+1)\Gamma(1+m-s)}\right.$$

$$\left.+\sum_{s=0}^{\infty}\frac{\left(\frac{z-1}{z+1}\right)^s\left[\psi(s+1)+\psi(s+m+1)-\psi(s-\nu)-\psi(s+m-\nu)\right]}{\Gamma(s+1)\Gamma(s+m+1)\Gamma(1+\nu-s)\Gamma(1+\nu-m-s)}\right\}$$

when $R(z) > 0$. The first sum is absent if $m = 0$

The case $\nu = n \geq m$ is of interest in connection with spheroidal harmonics. To evaluate $Q_n^m(z)$ we may write (46')

$$Q_\nu^m(z) = \frac{1}{2}P_\nu^m(z)\log\left(\frac{z+1}{z-1}\right)$$

$$+\frac{1}{2}\Gamma(\nu+1)\Gamma(\nu+m+1)\left(\frac{z+1}{2}\right)^{\nu}\left(\frac{z-1}{z+1}\right)^{\frac{m}{2}}\left\{\sum_{s=1}^{m}\frac{(-1)^s\left(\frac{z+1}{z-1}\right)^s\Gamma(s)}{\Gamma(s+\nu+1)\Gamma(s+\nu-m+1)\Gamma(1+m-s)}\right.$$

$$\left.+\sum_{s=0}^{\infty}\frac{\left(\frac{z-1}{z+1}\right)^s\left[\psi(s+1)+\psi(s+m+1)+\pi\cot\nu\pi-\psi(s-\nu)+\pi\cot\nu\pi-\psi(s+m-\nu)\right]}{\Gamma(s+1)\Gamma(s+m+1)\Gamma(1+\nu-s)\Gamma(1+\nu-m-s)}\right\}$$

Referring to 8_e I

$$\pi \cot \nu\pi - \psi(s-\nu) = -\left[\pi \cot(s-\nu)\pi + \psi(s-\nu)\right] = -\psi(1+\nu-s)$$
$$\pi \cot \nu\pi - \psi(s+m-\nu) \qquad = -\psi(1+\nu-m-s)$$

so that

$$Q_\nu^m(z) = \frac{1}{2} P_\nu^m(z) \log \frac{z+1}{z-1}$$

$$+\frac{1}{2}\Gamma(\nu+1)\Gamma(\nu+m+1)\left(\frac{z+1}{2}\right)^\nu\left(\frac{z-1}{z+1}\right)^{\frac{m}{2}}\left\{\sum_{s=1}^{m}\frac{(-1)^s\left(\frac{z+1}{z-1}\right)^s\Gamma(s)}{\Gamma(s+\nu+1)\Gamma(s+\nu-m+1)\Gamma(1+m-s)}\right.$$

$$\left.+\sum_{s=0}^{\infty}\frac{\left(\frac{z-1}{z+1}\right)^s\left[\psi(s+1)+\psi(s+m+1)-\psi(1+\nu-s)-\psi(1+\nu-m-s)\right]}{\Gamma(s+1)\Gamma(s+m+1)\Gamma(1+\nu-s)\Gamma(1+\nu-m-s)}\right\}$$

Reference to $(8)_g$ I shows that when $\nu \to n \geq m$ this infinite series vanishes for terms in which $s > n$

Hence if $n \geq m = 0, 1, 2 \cdots$ the following is valid everywhere

48) $$Q_n^m(z) = \frac{1}{2} P_n^m(z) \log\left(\frac{z+1}{z-1}\right) + S_n^m(z)$$

where S_n^m is a finite sum

49) $$S_n^m(z) \equiv \frac{1}{2}\left(\frac{z+1}{2}\right)^n\left(\frac{z-1}{z+1}\right)^{\frac{m}{2}} n!(n+m)! \cdot$$

$$\cdot\left\{\sum_{s=0}^{n-m}\frac{\left(\frac{z-1}{z+1}\right)^s\left[\psi(1+s)-\psi(1+n-s)+\psi(1+m+s)-\psi(1+n-m-s)\right]}{s!\,(s+m)!\,(n-s)!\,(n-m-s)!}\right.$$

$$-(-1)^{n-m}\sum_{s=n-m+1}^{n}\frac{(-1)^s\left(\frac{z-1}{z+1}\right)^s(s-n+m-1)!}{s!\,(s+m)!\,(n-s)!}$$

$$\left.+\sum_{s=1}^{m}\frac{(-1)^s\left(\frac{z+1}{z-1}\right)^s(s-1)!}{(s+m)!\,(s+n-m)!\,(m-s)!}\right\}$$

The last two sums are absent in the case $m = 0$.

4. Homographic transformations for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ converting hg. functions of z^2 into those of $\frac{1}{z^2}$, $1-z^2$, $\frac{1}{1-z^2}$, $\frac{z^2}{z^2-1}$ and $\frac{z^2-1}{z^2}$.

There are six expressions of this type for $P_\nu^\mu(z)$ and six for $Q_\nu^\mu(z)$. A sufficient number are tabulated below in order to obtain all by use of $(6)_a$ or $(6)_b$.

By Gauss's theorem applied to $A_\nu^\mu(z)$ and $B_\nu^\mu(z)$ these are continued into the interior of both loops of the lemniscate. fig 1 where $|1-z^2|<1$. The right-hand loop has an area common to the region of the plane to the right of the right branch of the hyperbola $x^2-y^2=\frac{1}{2}$, the two branches of which are the locus of $\left|\frac{z^2}{z^2-1}\right|=1$. They are the inversion of the lemniscate with respect to the circle $|z|=1$.

The polar equation of the lemniscate is $r^2=2\cos 2\theta$ where $z=re^{i\theta}$. The region of the plane where $\left|\frac{z^2}{z^2-1}\right|<1$ is between the two branches of the hyperbola, and the region where $\left|\frac{z^2-1}{z^2}\right|<1$ is to the right of the right branch and to the left of the left.

The schedule is similar to that of the preceding article 3, but a new consideration

arises here. The homographic transformations in that case were performed upon the variable $\frac{1-z}{2}$ which is a linear function of z so that the values of $F(\alpha,\beta,\gamma;z)$ upon one sheet of of its Riemann's surface were sufficient to give the values of $F(\alpha,\beta,\gamma;\frac{1-z}{2})$ upon one sheet of its Riemann's surface.

In the present case we start with the hg-functions of z^2 which define $A_\nu^\mu(z)$ and $B_\nu^\mu(z)$ in $(27)_a$ and $(27)_b$. The change of variable from z_1 to z by the eq $z_1 = z^2$ represents one sheet of the z_1-plane upon half of the z-plane, so that in general a knowledge of $F(\alpha,\beta,\gamma;z_1)$ upon two sheets of its z_1-Riemann's surface is necessary for a knowledge of $F(\alpha,\beta,\gamma;z^2)$ throughout the z-plane. The z-plane must be cut not only from $+1$ to ∞ but also from -1 to ∞ (taking the cuts both along the real axis). This is the cut that has been adopted for $I_\nu^\mu(z)$, $q_\nu^\mu(z)$ and $(1-z^2)^\nu$, and is the cut necessary for the analytic continuation of $A_\nu^\mu(z)$ and $B_\nu^\mu(z)$.

The part of the z-plane for which $R(z) > 0$ corresponds to the entire z_1-plane so that the transformations of the hg-functions (such as Gauss's or Euler's) which serve to give the

analytic continuation of $F(\alpha,\beta,\gamma,z_1)$ to the entire z_1-plane properly cut as in II, will be limited, when applied to functions of z^2, to the half-plane $R(z)>0$.

The results, however, may be extended to the entire z-plane without reference to the Riemann's surface by keeping in mind that $A_\nu^\mu(z)$ is an even function of z and $B_\nu^\mu(z)$ an odd one.

This fact shows that expression $(30)_a$ is valid inside either loop of the lemniscate, while $(30)_b$ for $B_\nu^\mu(z)$ is only correct inside the right loop. The application of Gauss's transformation to the hg-function in $(27)_b$ gives

$$\sin\mu\pi\, B_\nu^\mu(z) = z\left\{\frac{-\cos(\nu+\mu)\frac{\pi}{2}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\,F\left(\frac{\mu-\nu+1}{2},\frac{\mu+\nu}{2}+1,\mu+1;1-z^2\right)\right.$$
$$\left.+(1-z^2)^{-\mu}\frac{2^\mu\cos(\nu-\mu)\frac{\pi}{2}}{\Gamma(-\mu+1)}\,F\left(\frac{\nu-\mu}{2}+1,\frac{1-\nu-\mu}{2},-\mu+1;1-z^2\right)\right\} \tag{50}$$

which, being an odd function of z, is valid inside either loop of the lemniscate.

If we now apply to the first hg function on the right the fundamental formula (3) I, this becomes

$$(1-[1-z^2])^{-\frac{1}{2}}\,F\left(\frac{\mu-\nu}{2},\frac{\nu+\mu+1}{2},\mu+1;1-z^2\right)$$

where (since this theorem was derived by use of the

binomial series), the branch of the double-valued function $(1-[1-z^2])^{-\frac{1}{2}}$ is the one which reduces to $+1$ when $1-z^2=0$, that is, when $z=\pm 1$. Hence it is $\frac{1}{z}$ if $R(z)>0$. Similarly treating the second hg-function of (50) we obtain $(30)_b$ valid therefore in the right loop.

When z is inside the left loop of the lemniscate the expressions for $\mathcal{P}_\nu^\mu(z)$, $q_\nu^\mu(z)$, $P_\nu^\mu(z)$, and $Q_\nu^\mu(z)$ are obtained by using in $(28)_a$ $(28)_b$ $(29)_a$ and $(29)_b$ respectively the expression $(30)_a$ for $A_\nu^\mu(z)$ and for $B_\nu^\mu(z)$ either (50) or, the following

$$51)\quad \left.\begin{aligned} B_\nu^\mu(z) &= \phi(z^2) \text{ if } R(z)>0 \\ &= -\phi(z^2) \text{ if } R(z)<0 \end{aligned}\right\} |1-z^2|<1$$

where $\phi(z^2)$ is the even function of z in the second member of $(30)_b$. This makes B_ν^μ an odd function of z.

We thus find when z is inside the right loop

$$52)_a\quad P_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}}\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\,F\left(\frac{\mu-\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1, 1-z^2\right)$$

but inside the left loop we find

$$52)_b\quad P_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}}}{\sin\mu\pi}\left\{\frac{\sin\nu\pi\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\,F\left(\frac{\mu-\nu}{2}, \frac{\mu+\nu+1}{2}, \mu+1; 1-z^2\right)\right.$$

$$\left. -(1-z^2)^{-\mu}\frac{2^\mu\sin(\nu-\mu)\pi}{\Gamma(-\mu+1)}\,F\left(-\frac{\mu-\nu}{2}, \frac{\nu-\mu+1}{2}, -\mu+1; 1-z^2\right)\right\}$$

where $(1-z^2)^{-\mu} = (z^2-1)^{-\mu}e^{\pm i\mu\pi}$, the upper or lower sign

applying according as z is in the upper or lower half plane. The two equations $(52)_a$ and $(52)_b$ agree with $(33)_a$ and $(33)_b$ when $\nu-\mu$ is an integer, the factor $(z^2-1)^{\frac{\mu}{2}}$ in P_ν^μ being $(1-z^2)^{\frac{\mu}{2}}$ in T_ν^μ.

The foregoing formulas give the following relations which are useful in checking these and other analytic continuations.

$$53)_a \quad \left[\frac{P_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}}\right]_{z=0} = \frac{2^\mu\,\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\nu-\mu}{2}+1\right)}\cos(\nu-\mu)\frac{\pi}{2} = T_\nu^\mu(0)$$

$$53)_b \quad \left[\frac{P_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}}\right]_{z=+1} = \frac{\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}$$

$$53)_c \quad \left[\frac{P_\nu^\mu(z)}{(z^2-1)^{\frac{\mu}{2}}}\right]_{z=-1} = \frac{\sin\nu\pi}{\sin\mu\pi}\,\frac{\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)} \quad \text{if } R(\mu)<0$$

$$53)_d \quad \left[(z^2-1)^{\frac{\mu}{2}}P_\nu^\mu(z)\right]_{z=-1} = -\frac{2^\mu}{\pi}\sin(\nu-\mu)\pi\,\Gamma(\mu) \quad \text{if } R(\mu)>0$$

The left loop of the lemniscate has a finite area in common with the region of the cut, z-plane to the left of the left branch of the hyperbola. Hence in this common region, Euler's theorem may be applied to each of the hg-functions in the second member of $(52)_b$. The first one

is converted into an hg-function of $\frac{z^2-1}{z^2}$ having the factor $\left(1-[1-z^2]\right)^{-\left(\frac{\nu+\mu+1}{2}\right)} = \frac{1}{(z^2)^{\frac{\nu+\mu+1}{2}}}$ = the branch which

is $+1$ when $1-z^2=0$, which in this region can only be zero when $z = e^{\pm i\pi}$. Hence this factor is $\left(\frac{e^{\pm i\pi}}{z}\right)^{\nu+\mu+1}$ the upper or lower sign holding according as z is in the upper or lower half-plane.

This gives

$$54)\quad P_\nu^\mu(z) = \frac{1}{\sin\mu\pi}\left(\frac{e^{\pm i\pi}}{z}\right)^{\nu+1} \cdot \left\{\frac{e^{\pm i\mu\pi}\sin\mu\nu\pi\,\Gamma(\nu+\mu+1)}{2^\mu\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)}\left(1-\frac{1}{z^2}\right)^{\frac{\mu}{2}} F\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu}{2}+1, \mu+1; \frac{z^2-1}{z^2}\right)\right.$$
$$\left. - 2^\mu\frac{\sin(\nu-\mu)\pi}{\Gamma(-\mu+1)}\left(1-\frac{1}{z^2}\right)^{-\frac{\mu}{2}} F\left(\frac{\nu-\mu+1}{2}, \frac{\nu-\mu}{2}+1, -\mu+1; \frac{z^2-1}{z}\right)\right\}$$

which is valid at all points of the cut z-plane to the left of the left branch of the hyperbola.

The expression for $Q_\nu^\mu(z)$ in the same region is given by $(65)_b$ Eq (54) has the correct value when $z = e^{\pm i\pi}$. It also agrees with $(56)_a$ below when $z = re^{\pm i\pi}$ and $r \to \infty$.

In the region to the right of the right branch of the hyperbola we find by applying Euler's transformation to $(52)_a$

$$55)_a \quad P_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}}\,\Gamma(\nu+\mu+1)}{2^\mu z^{\nu+\mu+1}\,\Gamma(\nu-\mu+1)\,\Gamma(\mu+1)} \cdot F\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu}{2}+1, \mu+1; \frac{z^2-1}{z^2}\right)$$

and for the same region (as well as to left of the left branch)

$$55)_b \quad Q_\nu^\mu(z) = \frac{(z^2-1)^{\frac{\mu}{2}}\; 2^{\nu-1}}{z^{\nu+\mu+1}\,\sqrt{\pi}\,\Gamma(\nu-\mu+1)}\, g\left(\frac{\nu+\mu+1}{2}, \frac{\nu+\mu}{2}+1, \mu+1; \frac{z^2-1}{z^2}\right).$$

Gauss's transformation of $(55)_a$ gives

$$56)_a \quad P_\nu^\mu(z) = \frac{\sqrt{\pi}}{\cos\nu\pi}\left(1-\frac{1}{z^2}\right)^{\frac{\mu}{2}} \cdot$$

$$\cdot\left\{\frac{\sin(\nu-\mu)\pi}{\pi}\,\frac{\Gamma(\nu+\mu+1)}{(2z)^{\nu+1}\,\Gamma(\nu+\frac{3}{2})} F\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right)\right.$$

$$\left. + \frac{(2z)^\nu}{\Gamma(\nu-\mu+1)\,\Gamma(-\nu+\frac{1}{2})} \cdot F\left(\frac{\mu-\nu+1}{2}, \frac{\mu-\nu}{2}, -\nu+\frac{1}{2}; \frac{1}{z^2}\right)\right\}$$

$$= \frac{2^{\mu-1}(z^2-1)^{\frac{\mu}{2}}\sin(\nu-\mu)\pi}{z^{\mu+\nu+1}\,\pi^2\sin\nu\pi}\, g\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

and

$$56)_b \quad Q_\nu^\mu(z) = \frac{\sqrt{\pi}\,(z^2-1)^{\frac{\mu}{2}}\cos\mu\pi\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\, z^{\mu+\nu+1}\,\Gamma(\nu+\frac{3}{2})}\, F\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right)$$

$$= \frac{(z^2-1)^{\frac{\mu}{2}}\, 2^{\mu-1}\cos\mu\pi}{z^{\mu+\nu+1}}\, f\left(\frac{\mu+\nu+1}{2}, \frac{\mu+\nu}{2}+1, \nu+\frac{3}{2}; \frac{1}{z^2}\right).$$

Since both members of these equations are analytic functions of z at all points of the cut, z-plane which are exterior to the circle $|z|=1$, they are valid in this region.

Applying Euler's transformation to them gives

$$57)_a \quad P_\nu^\mu(z) = \frac{\sin(\nu-\mu)\pi}{\sqrt{\pi}\cos\nu\pi}\left\{\frac{(z^2-1)^{-\left(\frac{\nu+1}{2}\right)}\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma(\nu+\frac{3}{2})}F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}, \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)\right.$$

$$\left. - 2^\nu(z^2-1)^{\frac{\nu}{2}}\frac{\Gamma(\mu-\nu)}{\Gamma(-\nu+\frac{1}{2})}F\left(\frac{\mu-\nu}{2}, -\frac{\mu-\nu}{2}; -\nu+\frac{1}{2}; \frac{1}{1-z^2}\right)\right\}$$

$$57)_b \quad Q_\nu^\mu(z) = \frac{\sqrt{\pi}\cos\mu\pi\,\Gamma(\nu+\mu+1)}{2^{\nu+1}(z^2-1)^{\frac{\nu+1}{2}}\Gamma(\nu+\frac{3}{2})}F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}, \nu+\frac{3}{2}; \frac{1}{1-z^2}\right)$$

These are valid at all points of the cut, z-plane which are exterior to both loops of the lemniscate.

Finally for the region of the cut, plane between the two branches of the hyperbola we find by Gauss theorem on $(57)_b$

$$58) \quad Q_\nu^\mu(z) = \frac{2^{\mu-1}\sqrt{\pi}\cos\mu\pi}{(z^2-1)^{\frac{\nu+1}{2}}}\left\{\frac{\Gamma\left(\frac{\nu+\mu+1}{2}\right)}{\Gamma\left(\frac{\nu-\mu}{2}+1\right)}F\left(\frac{\nu+\mu+1}{2}, \frac{\nu-\mu+1}{2}, \frac{1}{2}; \frac{z^2}{z^2-1}\right)\right.$$

$$\left. - \frac{2z}{\sqrt{z^2-1}}\frac{\Gamma\left(\frac{\nu+\mu}{2}+1\right)}{\Gamma\left(\frac{\nu-\mu+1}{2}\right)}F\left(\frac{\nu+\mu}{2}+1, \frac{\nu-\mu}{2}+1, \frac{3}{2}; \frac{z^2}{z^2-1}\right)\right\}$$

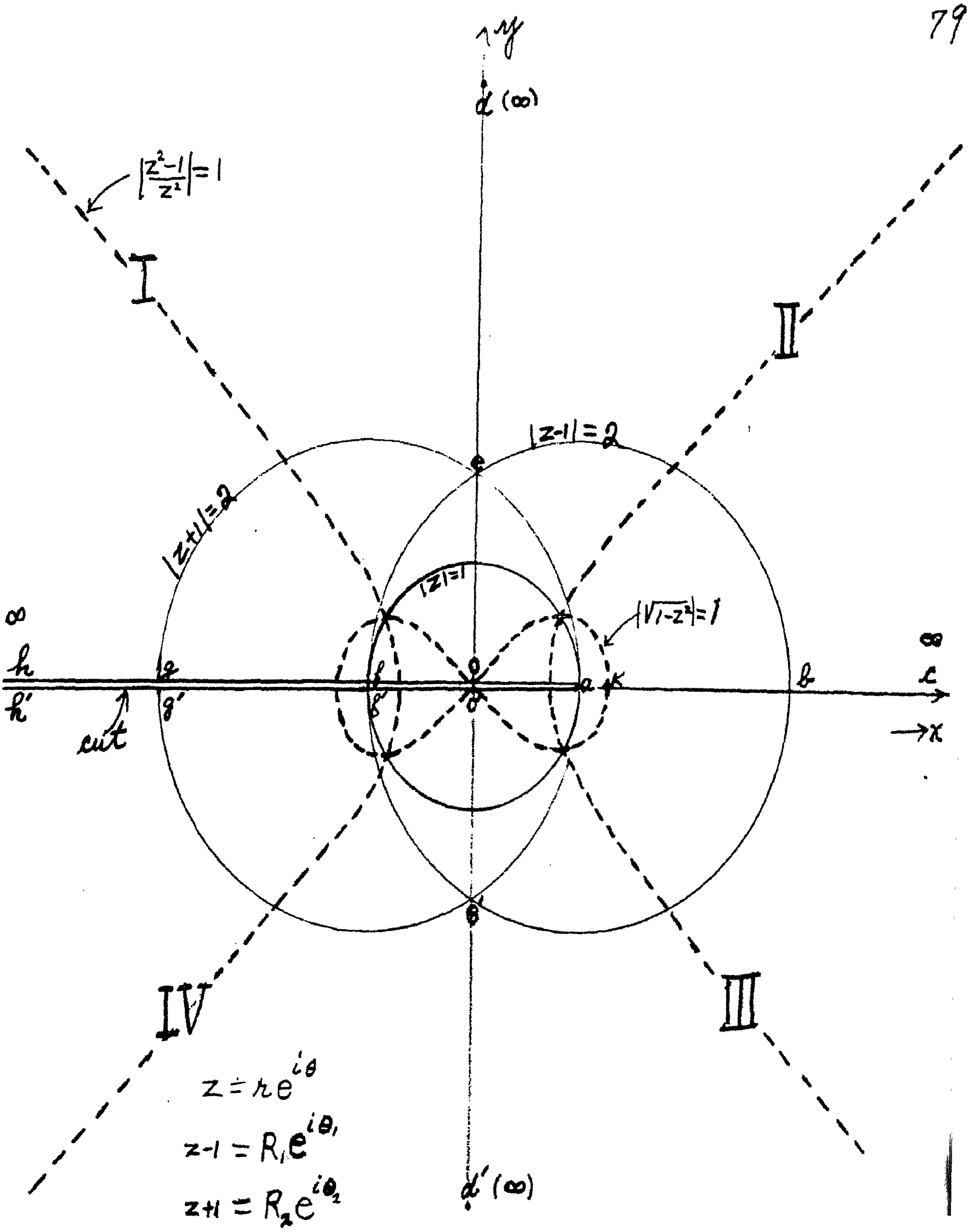


Fig 1. The z-plane cut along a f h for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$
The cut for $T_\nu^\mu(z)$ and $q_\nu^\mu(z)$ would be along hf and ac.

5.

(a) Expressions for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ valid in the entire cut, z-plane.

The complete boundary of the z-plane as cut for $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ consists of the infinite circle (the points $h\ d\ c\ d'\ h'$ of fig 1) together with both sides of the cut $h\,o\,a$ and $h'\,o'\,a$. The region thus bounded may be represented conformally upon the interior of a two-sheeted circle of the t-plane defined by $t=\rho e^{i\phi}$, $0<\rho<1$, $-2\pi<\phi<2\pi$ which is cut along its radius from $t=0$ to $t=1$.

The equation of transformation

59)$_a$ $$z=\frac{t+1}{2\sqrt{t}} \quad \text{or} \quad x=\frac{1}{2}\left[\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\right]\cos\frac{\phi}{2},\ y=-\frac{1}{2}\left[\frac{1}{\sqrt{\rho}}-\sqrt{\rho}\right]\sin\frac{\phi}{2},$$

is equivalent to

59)$_b$ $$\sqrt{z^2-1}=\frac{1-t}{2\sqrt{t}}$$

so that

59)$_c$ $$t=\frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}.$$

If $z-1=R_1e^{i\theta_1}$ and $z+1=R_2e^{i\theta_2}$ then $\sqrt{z^2-1}=\sqrt{R_1R_2}\,e^{i\frac{(\theta_1+\theta_2)}{2}}$ and $\arg\sqrt{z^2-1}$, $(\equiv \frac{\theta_1+\theta_2}{2})$, increases from $-\pi$ to π as the point z moves from the lower side of the cut in quadrant IV of fig 1 to the upper side of the cut in quadrant I. Hence $\arg\sqrt{z^2-1}$ will be

uniquely determined by the two equations, which are equivalent to (59).

60)$_a$ $\sqrt{R_1R_2}\cos\frac{\theta_1+\theta_2}{2} = \frac{1}{2}\left(\frac{1}{\sqrt{\rho}}-\sqrt{\rho}\right)\cos\frac{\phi}{2}$

60)$_b$ $\sqrt{R_1R_2}\sin\frac{\theta_1+\theta_2}{2} = -\frac{1}{2}\left(\frac{1}{\sqrt{\rho}}+\sqrt{\rho}\right)\sin\frac{\phi}{2}$

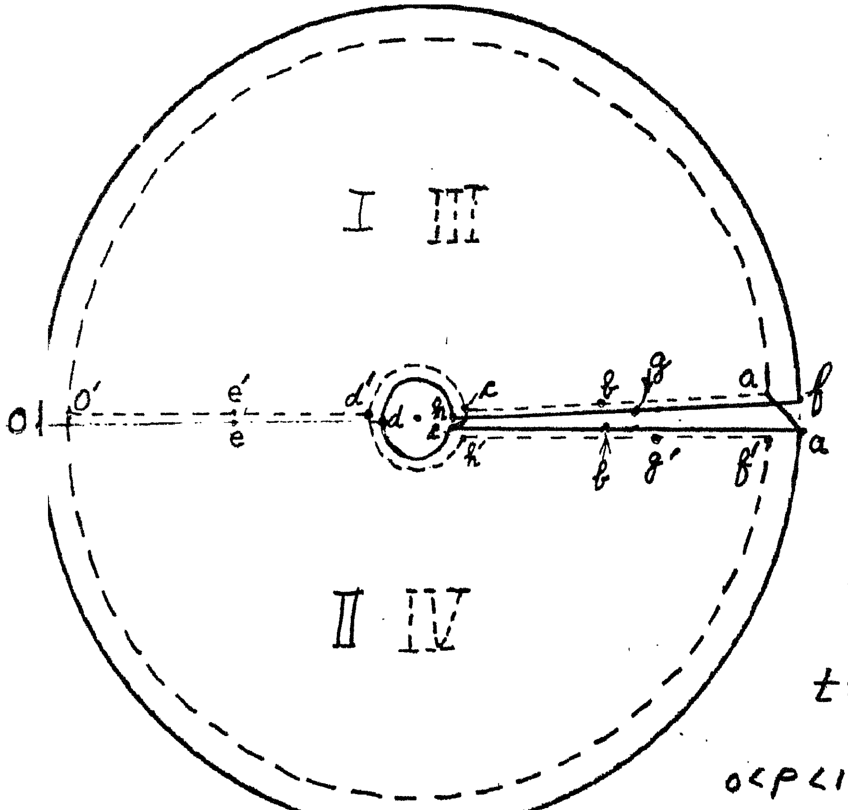
together with the conditions

60)$_c$ $-\pi < \frac{\theta_1+\theta_2}{2} \equiv \arg\sqrt{z^2-1} = \arg\frac{1-t}{2\sqrt{t}} < \pi$

quadrant.

60)$_d$ $-\pi < \frac{\phi}{2} < \pi$ { dotted lower sheet $0 < \phi < 2\pi$ Quadrant III & IV }

The u... circle indicated by h d c d' h' (but not drawn in fig 1) corresponds to two infinitesimal circles in fig 2, their centers being at the origin $t=0$. The dotted one is supposed to be on the lower sheet. The upper side of the part f o a of fig 1 is the circumference of the unit circle in fig 2 (on the upper sheet). The lower part f' o a is the dotted circle on the lower sheet. The line a b c ($\phi=0$) of fig 1 is not a cut or barrier and the two radii marked a b c in fig 2 are supposed to be identical, being common to both sheets. as indicated by the single connection at a a and c c. The radial lines of fig 2 marked f g h ($\phi=-2\pi$) and f' g' h' ($\phi=2\pi$) are not connected. They are barriers as required by fig 1



$t = \rho e^{i\phi}$

$0 < \rho < 1, \quad -2\pi < \phi < 2\pi$

Fig 2

$\phi = 0$ is abc, $\phi = 2\pi$ is $f'g'h'$, $\phi = -2\pi$ is fgh

The two-sheeted circle of t variable where $t = \dfrac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}$ or $z = \dfrac{t+1}{2\sqrt{t}}$

Replacing z by $\frac{1}{z^2}$ in formula $(10)_a$ III gives the following equivalent of $(56)_b$

$$61)_a \quad Q_\nu^\mu(z) = \frac{2^\mu \sqrt{\pi} \cos\mu\pi \, (z^2-1)^{\frac{\mu}{2}} \, \Gamma(\nu+\mu+1)}{[z+\sqrt{z^2-1}]^{\mu+\nu+1} \, \Gamma(\nu+\frac{3}{2})} F\left(\mu+\tfrac{1}{2}, \mu+\nu+1, \nu+\tfrac{3}{2}; \frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right)$$

which is valid at all points of the cut z-plane, as is

$$61)_b \quad P_\nu^\mu(z) = \frac{\sin(\nu-\mu)\pi \, . \, 2^\mu (z^2-1)^{\frac{\mu}{2}}}{\sqrt{\pi} \cos\nu\pi \, (z+\sqrt{z^2-1})^\mu} \left\{ \frac{(z+\sqrt{z^2-1})^{-\nu-1}}{\Gamma(\mu+\frac{1}{2})} F\left(\mu+\tfrac{1}{2}, \mu+\nu+1, \nu+\tfrac{3}{2}; \frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) \right.$$

$$\left. - \frac{(z+\sqrt{z^2-1})^{\nu}}{\Gamma(\mu+\frac{1}{2})} F\left(\mu+\tfrac{1}{2}, \mu-\nu; -\nu+\tfrac{1}{2}; \frac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}\right) \right\}$$

(b) Whipple's Relations

The right half of the z-plane cut from zero to 1 is represented upon the right half of the z'-plane similarly cut (and conversely, since the relation is symmetrical,) by the equation

$z = \frac{z'}{\sqrt{z'^2-1}}$. The lemniscate transforms into itself since $(z^2-1)(z'^2-1)=1$. The conformance is indicated by similar lettering in fig 1 and fig 3. Comparison of eq (55)a and (56)b shows that

62)a $\quad Q_\nu^\mu(z) = \sqrt{\frac{\pi}{2}}\,(z^2-1)^{-\frac14}\cos\mu\pi\,\Gamma(\mu-\nu)\,P_{\mu-\frac12}^{\nu+\frac12}\left(\frac{z}{\sqrt{z^2-1}}\right)$

which may be inverted (or by using (6)b) into

62)b $\quad P_\nu^\mu(z) = \sqrt{\frac{2}{\pi}}\,(z^2-1)^{-\frac14}\,\frac{\sin(\nu-\mu)\pi}{\pi\sin\nu\pi}\,\Gamma(\mu-\nu)\,Q_{\mu-\frac12}^{\nu+\frac12}\left(\frac{z}{\sqrt{z^2-1}}\right)$

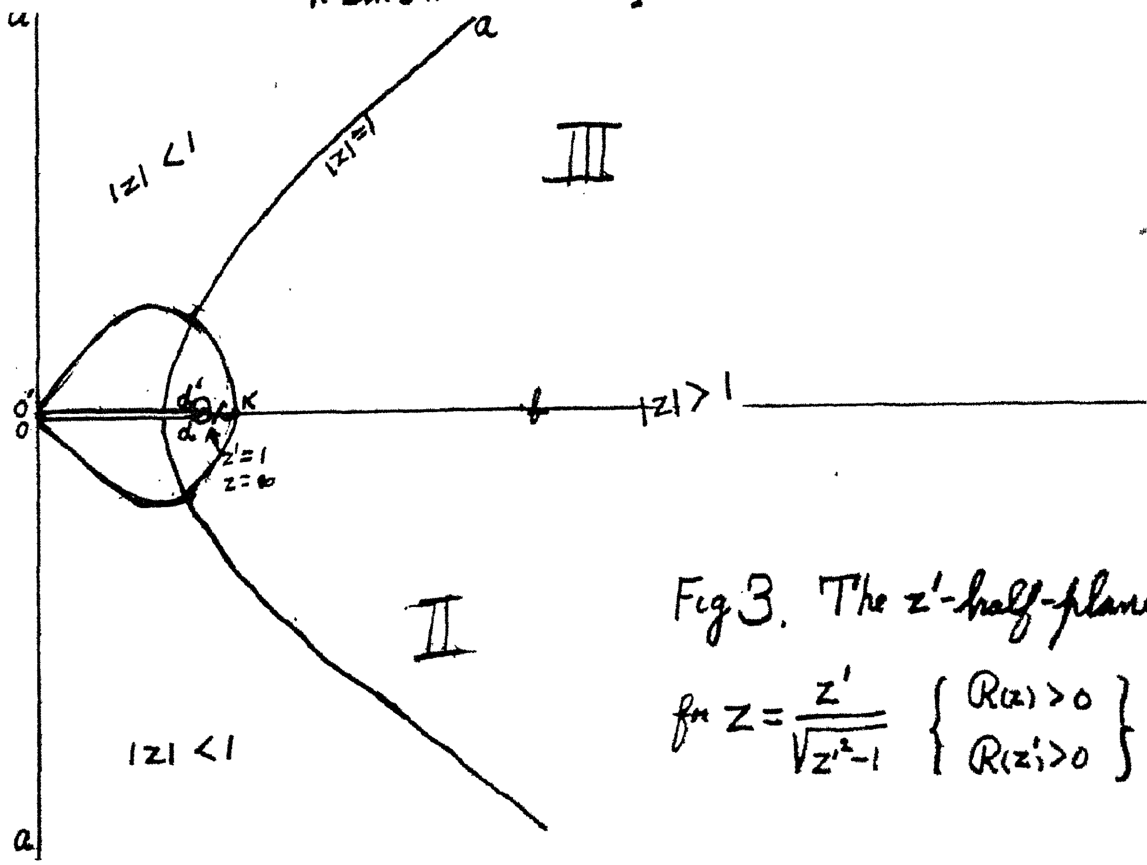


Fig 3. The z'-half-plane for $z = \frac{z'}{\sqrt{z'^2-1}}$ $\left\{\begin{matrix} R(z)>0 \\ R(z')>0 \end{matrix}\right\}$

When x is a positive real, Whipple's transformation gives the following

62)$_c$
$$\Gamma(\tfrac{1}{2}+\nu-\mu)\, T^{\mu}_{\nu-\frac{1}{2}}(\tanh x) = \sqrt{\frac{2\cosh x}{\pi}}\; \frac{e^{i(\frac{1}{2}+\mu)\frac{\pi}{2}}}{\cos\nu\pi}\cdot Q^{\nu}_{\mu-\frac{1}{2}}(i\sinh x)$$

and

$$\Gamma(\tfrac{1}{2}+\nu-\mu)\, T^{\mu}_{\nu-\frac{1}{2}}(-\tanh x) =$$

$$= \sqrt{\frac{2\cosh x}{\pi}}\cdot e^{-i(\frac{1}{2}+\mu)\frac{\pi}{2}} \left\{ \frac{e^{i\nu\pi}}{\cos\nu\pi}\cdot Q^{\nu}_{\mu-\frac{1}{2}}(i\sinh x) + i\pi\, P^{\nu}_{\mu-\frac{1}{2}}(i\sinh x) \right\}$$

62)$_d$
$$\Gamma(\tfrac{1}{2}+\nu-\mu)\, P^{\mu}_{\nu-\frac{1}{2}}(\coth x) = \sqrt{\frac{2\sinh x}{\pi}}\; \frac{Q^{\nu}_{\mu-\frac{1}{2}}(\cosh x)}{\cos\nu\pi}$$

$$Q^{\mu}_{\nu-\frac{1}{2}}(\coth x) = \cos\mu\pi \sqrt{\frac{\pi\sinh x}{2}}\, \Gamma(\tfrac{1}{2}-\nu+\mu)\, P^{\nu}_{\mu-\frac{1}{2}}(\cosh x)$$

If $0<\alpha<\frac{\pi}{2}$

62)$_e$
$$\sqrt{\frac{\pi\sin\alpha}{2}}\, \Gamma(\tfrac{1}{2}+\mu-\nu)\, T^{\nu}_{\mu-\frac{1}{2}}(\cos\alpha) = \frac{e^{i(\frac{1}{2}+\nu)\frac{\pi}{2}}}{\cos\mu\pi}\, Q^{\mu}_{\nu-\frac{1}{2}}(i\cot\alpha)$$

$$\sqrt{\frac{\pi\sin\alpha}{2}}\, \Gamma(\tfrac{1}{2}+\mu-\nu)\, T^{\nu}_{\mu-\frac{1}{2}}(-\cos\alpha) =$$

$$= e^{-i(\frac{1}{2}+\nu)\frac{\pi}{2}} \left[\frac{e^{i\mu\pi}}{\cos\mu\pi}\, Q^{\mu}_{\nu-\frac{1}{2}}(i\cot\alpha) + i\pi\, P^{\mu}_{\nu-\frac{1}{2}}(i\cot\alpha) \right]$$

6. $P_\nu^\mu(\cos\omega)$ and $Q_\nu^\mu(\cos\omega)$ and Laplace's Integrals.

If $\omega = \alpha + i\beta$, the points on the two-sheeted Riemann's surface of the t variable are put into (1,1) correspondence with the points of the semi-infinite strip of the ω-plane, $-\pi < \alpha < \pi$, $0 < \beta < \infty$, by the transformation

63) $\quad t \equiv \rho e^{i\phi} = e^{2i\omega} \quad (\rho = e^{-2\beta}$ and $\phi = 2\alpha)$.

Hence the points in this strip are in (1,1) correspondence with the points in the cut, z-plane, by the equation

64)$_a$ $\quad z = \cos\omega$ or $x = \cos\alpha\cosh\beta$, $y = -\sin\alpha\sinh\beta$

so that

64)$_b$ $\quad e^{i\omega} = z - \sqrt{z^2-1}$ or $\left\{\begin{array}{l} \dfrac{x^2}{\cosh^2\beta} + \dfrac{y^2}{\sinh^2\beta} = 1 \\ \dfrac{x^2}{\cos^2\alpha} - \dfrac{y^2}{\sin^2\alpha} = 1 \end{array}\right\}$

In accordance with (59)$_b$ we must take $\sqrt{z^2-1} = e^{-\frac{i\pi}{2}}\sin\omega$, that is,

64)$_c$ $\left\{\begin{array}{l} \arg\sin\omega = \frac{\pi}{2} + \arg\sqrt{z^2-1} = \frac{\pi}{2} + \frac{\theta_1+\theta_2}{2} \quad \left(\text{so that } \frac{-\pi}{2} < \arg\sin\omega < \frac{3\pi}{2}\right) \\ (z^2-1)^{\mu/2} = e^{-\frac{i\mu\pi}{2}}\sin^\mu\omega \end{array}\right.$

The correspondence between the z-plane and the ω-strip is shown by similar lettering in figures 1 and 4. The infinite circle of the z-plane $h\,d\,c\,d'\,h'$ corresponds to $\beta = +\infty$.

$$65)_a \quad Q_\nu^\mu(\cos\omega) = \sqrt{\pi}\cos\mu\pi\,(1-e^{2i\omega})^\mu e^{(\nu+1)i\omega}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\frac{3}{2}+\nu)}F(\mu+\tfrac{1}{2},\mu+\nu+1,3/2+\nu;e^{2i\omega})$$

and

$$65)_b \quad P_\nu^\mu(\cos\omega) = \frac{(1-e^{2i\omega})^\mu \sin(\nu-\mu)\pi}{\sqrt{\pi}\cos\nu\pi}\left\{e^{(\nu+1)i\omega}\frac{\Gamma(\mu+\nu+1)}{\Gamma(\frac{3}{2}+\nu)}F(\mu+\tfrac{1}{2},\mu+\nu+1,\tfrac{3}{2}+\nu;e^{2i\omega}) - e^{-\nu i\omega}\frac{\Gamma(\mu-\nu)}{\Gamma(\frac{1}{2}-\nu)}F(\mu+\tfrac{1}{2},\mu-\nu,\tfrac{1}{2}-\nu;e^{2i\omega})\right\}$$

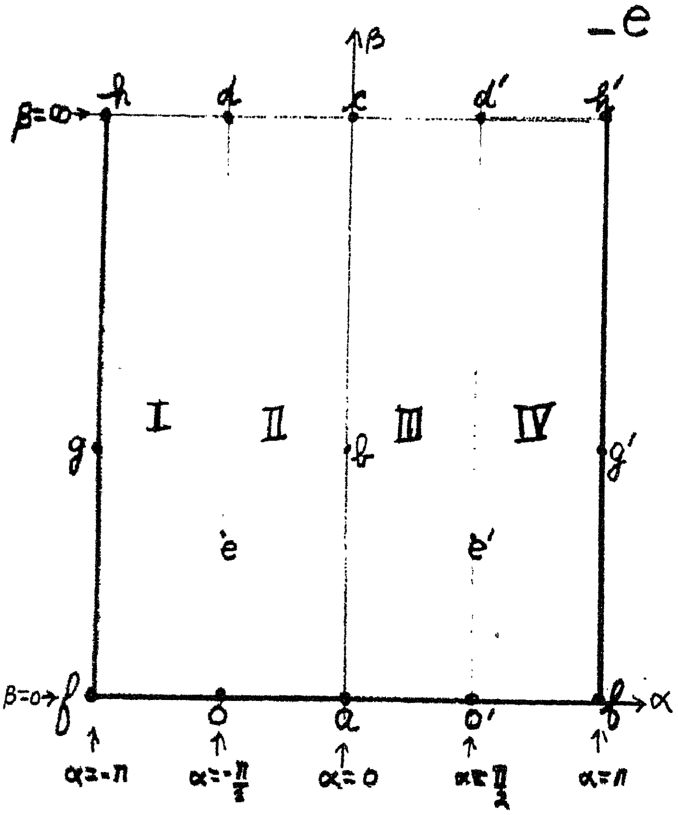


Fig 4 The ω-strip $\omega=\alpha+i\beta$, $z=\cos\omega$

Also in III and IV

$$(65)_c \quad T_\nu^\mu(\cos\omega) = e^{\frac{i\mu\pi}{2}}P_\nu^\mu(\cos\omega)$$

which holds (if $(65)_b$ be used) for $0<\alpha<\pi$ and $-\infty<\beta<\infty$ since the z-plane as cut for $T_\nu^\mu(z)$ is represented on this endless strip.

To obtain Laplace's integrals we write

$$(1-2h\cos\omega+h^2)^\mu = (1-he^{i\omega})^\mu\cdot(1-he^{-i\omega})^\mu.$$

Each factor may be expanded in a

binomial series and their product will be an absolutely convergent double series if

66) $|h|<1$ and $-\beta_0 < \beta < \beta_0 \equiv \log \frac{1}{|h|}$

The arrangement of this double series in ascending powers of h gives (for $|h|<1$ and $0 \le \beta < \log \frac{1}{|h|}$)

$$67)_a \quad (1-2h\cos\omega+h^2)^{\mu} = \frac{2^{\mu+\frac{1}{2}}\sqrt{\pi}\, e^{-i\frac{\pi}{2}(\mu+\frac{1}{2})} \sin^{\mu+\frac{1}{2}}\omega}{\Gamma(-\mu)} \sum_{s=0}^{\infty} h^s P^{-\mu-\frac{1}{2}}_{-\mu-\frac{1}{2}+s}(\cos\omega)$$

which may be written (for $|h|<1$ and z inside the ellipse $(64)_b$ $\beta=\beta_0=\log\frac{1}{|h|}$)

$$67)_b \quad \frac{(z^2-1)^{\frac{\mu}{2}}}{(1-2hz+h^2)^{\mu+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^{\mu}\,\Gamma(\mu+\frac{1}{2})} \sum_{s=0}^{\infty} h^s P^{\mu}_{\mu+s}(z)$$

The double-series may also be arranged in powers of $e^{i\omega}$. This gives (replacing ω by ω') the Fourier's series

$$68) \quad (1-2h\cos\omega'+h^2)^{\nu} = \frac{2}{\Gamma(-\nu)} \sum_{m=0}^{\infty} \epsilon_m h^m \frac{\Gamma(m-\nu)}{\Gamma(m+1)} F(-\nu, m-\nu, m+1; h^2) \cos m\omega'$$

where $\epsilon_0 = \frac{1}{2}$, $\epsilon_m = 1$ for $m \neq 0$

provided that $|h|<1$ and $-\beta_0 < \beta' < \beta_0$.

If we let

$$69)_a \quad h = i\tan\frac{\omega}{2} = e^{i\pi}\sqrt{\frac{z-1}{z+1}} = \sqrt{\frac{R_1}{R_2}}\, e^{i(\pi+\frac{\theta_1-\theta_2}{2})} = \frac{i\sin\omega}{1+\cos\omega} \quad \text{then}$$

$$69)_b \quad |h| = \sqrt{\frac{R_1}{R_2}} = \sqrt{\frac{\cosh\beta-\cos\alpha}{\cosh\beta+\cos\alpha}}$$ Eq $(64)_c$ requires that

69)$_c$ $\arg \tan\frac{\omega}{2} = \frac{\theta_1-\theta_2}{2} + \frac{\pi}{2}$ so $\arg h = \frac{\theta_1-\theta_2}{2} + \pi$

Then in order that $|h|<1$ the point ω must be in region II + III of fig 3 which corresponds to $R(z)>0$ in fig 1. In that case β_0 is a function of (x,y) or (α,β) given by

69)$_d$ $$\beta_0(\alpha,\beta) = \frac{1}{2}\log\frac{R_2}{R_1} = \frac{1}{2}\log\left(\frac{\cosh\beta+\cos\alpha}{\cosh\beta-\cos\alpha}\right) > 0$$

and

69)$_e$ $$1-2h\cos\omega'+h^2 = 1-2i\tan\frac{\omega}{2}\cos\omega' + \left(i\tan\frac{\omega}{2}\right)^2 =$$

$$= \frac{\cos\omega - i\sin\omega\cos\omega'}{\cos^2\frac{\omega}{2}} = \frac{z+\sqrt{z^2-1}\cdot z'}{\left(\frac{z+1}{2}\right)}$$

Since $h^m = (-1)^m\left(\frac{z-1}{z+1}\right)^{\frac{m}{2}}$ the hg function in (68) is seen by reference to (45) to be given by

$$\frac{h^m\,\Gamma(m-\nu)}{\Gamma(-\nu)\,\Gamma(m+1)} F(-\nu, m-\nu, m+1; h^2) = \left(\frac{2}{z+1}\right)^{\nu}\frac{\Gamma(\nu+1)}{\Gamma(\nu+m+1)} P_\nu^m(z)$$

so that (68) becomes

70) $$(z+\sqrt{z^2-1}\cos\omega')^{\nu} = (\cos\omega - i\sin\omega\cos\omega')^{\nu} = 2\,\Gamma(\nu+1)\sum_{m=0}^{\infty}\epsilon_m\frac{P_\nu^m(\cos\omega)}{\Gamma(\nu+m+1)}\cos m\omega'$$

where $\omega = \alpha + i\beta$ and $\omega' = \alpha' + i\beta'$.

This is valid in region II + III where $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$, $0 \le \beta \le \infty$ provided also that $-\beta_0 < \beta' < \beta_0$. where $\beta_0(\alpha,\beta)$ is given by (69)$_d$.

If $\beta = 0$, $\beta_0 = \log|\cot\frac{\alpha}{2}|$ and in general, for the geometric meaning of this second condition, we may picture a real surface above the ω-strip of fig 3, the height

above the plane of the paper at any point (α, β) being the positive real β_0 given by $(69)_d$. The contours $\beta_0 =$ constant on this surface correspond to the circles of the z-plane $\frac{R_2}{R_1} = e^{2\beta_0}$, that is

$$(x - \coth 2\beta_0)^2 + y^2 = \frac{1}{\sinh^2 2\beta_0}$$

In eq(68) $\omega' = \alpha' + i\beta'$ where α' may have any real value. We obtain Laplace's first integral by multiplying (70) by $\cos m\omega' . d\omega'$ and integrating along any path from $\beta' - \pi$ to $\beta' + \pi$ which is equivalent to the straight line $\beta' =$ constant $< \beta_0$. This path may be taken as the line $\beta' = 0$. Hence the following are valid if $-\frac{\pi}{2} < \alpha < \frac{\pi}{2}$ i.e. $R(z) > 0$.

$71)_a$
$$P_\nu^m(z) = P_\nu^m(\cos\omega) = \frac{\Gamma(m+\nu+1)}{2\pi\,\Gamma(\nu+1)} \int_{\beta'-\pi}^{\beta'+\pi} (z + \sqrt{z^2-1}\cos\omega')^\nu \cos m\omega'\, d\omega'$$

$$= \frac{\Gamma(m+\nu+1)}{2\pi\,\Gamma(\nu+1)} \int_{\beta'-\pi}^{\beta'+\pi} (\cos\omega - i\sin\omega\cos\omega')^\nu \cos m\omega'\, d\omega'$$

Or since by $(8)_a$ $P_\nu^m(z) = P_{-\nu-1}^m(z)$, m being an integer, this gives Laplace's second integral

$71)_b$
$$P_\nu^m(z) = P_\nu^m(\cos\omega) = \frac{(-1)^m\,\Gamma(\nu+1)}{2\pi\,\Gamma(\nu-m+1)} \int_{\beta'-\pi}^{\beta'+\pi} \frac{\cos m\omega'\, d\omega'}{(z + \sqrt{z^2-1}\cos\omega')^{\nu+1}}$$

$$= \frac{(-1)^m\,\Gamma(\nu+1)}{2\pi\,\Gamma(\nu-m+1)} \int_{\beta'-\pi}^{\beta'+\pi} \frac{\cos m\omega'\, d\omega'}{(\cos\omega - i\sin\omega\cos\omega')^{\nu+1}}$$

If $R(z_1)>0$ and $R(z_2)>0$ then by (70) (if ϕ and α' are real)

$$\left[z_1+\sqrt{z_1^2-1}\cos(\phi-\alpha')\right]^{\nu}=2\Gamma(\nu+1)\sum_{m=0}^{\infty}\epsilon_m\frac{P_\nu^{-m}(z_1)}{\Gamma(\nu+m+1)}\cos m(\phi-\alpha')$$

and

$$\left[z_2+\sqrt{z_2^2-1}\cos\alpha'\right]^{-\nu-1}=\frac{2}{\Gamma(\nu+1)}\sum_{s=0}^{\infty}\epsilon_s(-1)^s\,\Gamma(\nu-s+1)P_\nu^{s}(z_2)\cos s\alpha'$$

Multiplying these together and integrating with respect to α' gives

72) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\left[z_1+\sqrt{z_1^2-1}\cos(\phi-\alpha')\right]^{\nu}}{\left[z_2+\sqrt{z_2^2-1}\cos\alpha'\right]^{\nu+1}}d\alpha'=2\sum_{m=0}^{\infty}\epsilon_m(-1)^m\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}P_\nu^{m}(z_1)P_\nu^{-m}(z_2)\cos m\phi$$

It will be shown in the following article that this integral is $P_\nu(z)$ where

72)$_a$ $$z=z_1z_2-\sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\phi$$

The result is the so-called addition theorem for $P_\nu(z)$. Hobson, "Spherical Harmonics", p 259 gives a generalization of Heine's integral which in the notation used here is

73)$_a$ $$Q_\nu^{\mu}(z)=\cos\mu\pi\frac{\Gamma(\nu+1)}{\Gamma(\nu-\mu+1)}\int_0^{\infty}\frac{\cosh\mu\beta'\,d\beta'}{\left[z+\sqrt{z^2-1}\cosh\beta'\right]^{\nu+1}}\quad\text{if}\begin{pmatrix}R(\nu+\mu+1)>0\\R(\nu-\mu+1)>0\end{pmatrix}$$

for any z in the cut z-plane.

The expansion (70) is valid if $R(z) > 0$. Hence we may apply Whipple's transformation (62)$_b$. To do this we first replace z by z_1 in (70) then ν by $\mu - \frac{1}{2}$ and ω' by $\omega' + \pi$ (since (70) is valid for any real value of α'). This gives

$$(z_1 - \sqrt{z_1^2 - 1}\cos\omega')^{\mu - \frac{1}{2}} = 2\Gamma(\mu + \tfrac{1}{2}) \sum_{m=0}^{\infty} (-1)^m \epsilon_m \frac{P^{m}_{\mu-\frac{1}{2}}(z_1)}{\Gamma(m + \frac{1}{2} + \mu)} \cos m\omega'$$

If we now let $z_1 = \dfrac{z}{\sqrt{z^2-1}}$ where $R(z_1) > 0$ $R(z) > 0$, then by (61)$_b$ this becomes

74)$_b$ $$(z - \cos\omega')^{\mu - \frac{1}{2}} = \left(\frac{2}{\pi}\right)^{3/2} (z^2-1)^{\frac{\mu}{2}} \Gamma(\mu + \tfrac{1}{2}) \sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma(m + \frac{1}{2} - \mu)}{\Gamma(m + \frac{1}{2} + \mu)} Q^{\mu}_{m-\frac{1}{2}}(z) \cos m\omega'$$

which holds for all values of z in the half-plane $R z > 0$, cut from 0 to 1 as in fig 2, and for all real values of α' where $\omega' = \alpha' + i\beta'$, and for $\beta' = 0$ or in general $0 < \beta' < \beta_0$ where β_0 depends upon z as in (69)$_d$, ($z = \cos(\alpha + i\beta)$).

From (74) we find for $R(z) > 0$

73)$_c$ $$Q^{\mu}_{m-\frac{1}{2}}(z) = Q^{\mu}_{-m-\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}} \frac{\Gamma(m + \frac{1}{2} + \mu)}{2(z^2-1)^{\mu/2}\Gamma(\mu + \frac{1}{2})\Gamma(m + \frac{1}{2} - \mu)} \int_{\beta' - \pi}^{\beta' + \pi} (z - \cos\omega')^{\mu - \frac{1}{2}} \cos m\omega' \, d\omega'$$

and since $Q^{\mu}_{m-\frac{1}{2}}(z) = \dfrac{\Gamma(m + \frac{1}{2} + \mu)}{\Gamma(m + \frac{1}{2} - \mu)} Q^{-\mu}_{m-\frac{1}{2}}(z)$.

73)$_d$ $$Q^{\mu}_{m-\frac{1}{2}}(z) = \frac{\cos\mu\pi\, \Gamma(\mu + \frac{1}{2})(z^2-1)^{\frac{\mu}{2}}}{2\sqrt{2\pi}} \int_{\beta' - \pi}^{\beta' + \pi} \frac{\cos m\omega' \, d\omega'}{(z - \cos\omega')^{\mu + \frac{1}{2}}}$$

To change the variable of integration ω' to z' by (64) in the equations (71) or $(73)_{c,d}$ we make use eq $(6)_e$ I. By applying Euler's Theorem to it we obtain when $z' = \cos\omega'$,

$$\cos m\omega' = z'^m F\left(-\frac{m}{2}, -\frac{m}{2}+\frac{1}{2}, \frac{1}{2}; \frac{z'^2-1}{z'^2}\right) = \sqrt{\frac{\pi}{2}}(z'^2-1)^{\frac{1}{4}} \cdot m P^{-\frac{1}{2}}_{m-\frac{1}{2}}(z')$$

by $(55)_a$ (This is $+1$ when $m=0$.)

Eqs $(73)_c$ and $(73)_d$ transform into

$$73)_e \quad Q^{\mu}_{m-\frac{1}{2}}(z) = \frac{\pi(z^2-1)^{-\frac{\mu}{2}}\Gamma(m+\frac{1}{2}+\mu)}{\Gamma(m+\frac{1}{2}-\mu)\Gamma(\mu+\frac{1}{2})} \cdot \frac{m}{4i}\int (z'^2-1)^{-\frac{1}{4}} P^{-\frac{1}{2}}_{m-\frac{1}{2}}(z') \frac{dz'}{(z-z')^{\frac{1}{2}-\mu}}$$

$$73)_f \quad Q^{\mu}_{m-\frac{1}{2}}(z) = (z^2-1)^{\frac{\mu}{2}}\cos\mu\pi\,\Gamma(\mu+\tfrac{1}{2})\ \frac{m}{4i}\int (z'^2-1)^{-\frac{1}{4}} P^{-\frac{1}{2}}_{m-\frac{1}{2}}(z') \frac{dz'}{(z-z')^{\frac{1}{2}+\mu}}$$

where the point z lies outside the path, which in each integral is an ellipse with foci at ± 1, which begins on the lower side of the cut on the negative real axis and ends on the upper side.

When $\mu = 0$ these become

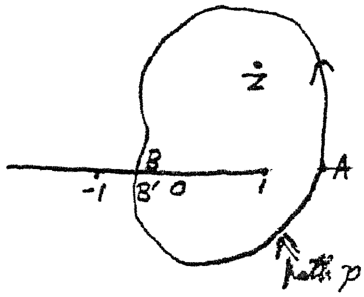
$$73)_g \quad Q_{m-\frac{1}{2}}(z) = \frac{1}{2i\sqrt{2}}\int \frac{dz'}{\sqrt{(z'^2-1)(z-z')}} = \frac{1}{2\sqrt{2}}\int_{-\pi}^{\pi} \frac{\cos m\alpha'}{\sqrt{z-\cos\alpha'}}\,d\alpha'$$

7. Schläfli's Integrals for $P_\nu^m(z)$ and $Q_\nu^m(z)$

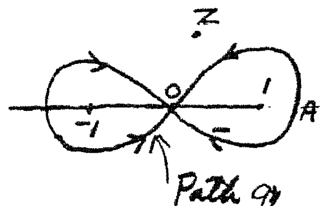
where $m = 0, 1, 2, 3, \ldots$

These are

74)$_a$ $$P_\nu^m(z) = \frac{(z^2-1)^{\frac{m}{2}}\,\Gamma(m+\nu+1)}{2^\nu\,\Gamma(\nu+1)}\cdot\frac{1}{2\pi i}\int_p \frac{(z'^2-1)^\nu\,dz'}{(z'-z)^{\nu+m+1}}$$



74)$_b$ $$Q_\nu^m(z) = \frac{(z^2-1)^{\frac{m}{2}}(-1)^m\,\Gamma(m+\nu+1)}{2^{\nu+1}\sin\nu\pi\,\Gamma(\nu+1)}\,\frac{1}{2i}\int_q \frac{(z'^2-1)^\nu\,dz'}{(z'-z)^{\nu+m+1}}$$



In both of these the z-plane is cut from $-\infty$ to $+1$ along the real axis. In each the integrand is a function of z' which has branch points which the path encircles starting from an initial point A say on the positive real axis to the right of $+1$ and returning to A. In this method of Cauchy the integrand is not a function of the position only of z' but depends upon the path travelled in reaching z', starting from the point A where $\arg(z-1)$ and $\arg(z+1)$ vanish, and $(\arg z'-z)_{z'=A}$ is the branch between $-\pi$ and π.

The path p encircles the fixed points z and $+1$ but not -1. Writing the integrand of (74)$_a$ $(z'+1)^\nu \frac{1}{(z'-z)^{m+1}}\cdot\left(\frac{z'-1}{z'-z}\right)^\nu$ the first factor is unaltered by description of the path p, as also the second since in

is an integer, and also the third since $\arg(z'-1)$ and $\arg(z'-z)$ each increase by 2π in this description. Hence $(74)_a$ defines $P_\nu^\mu(z)$ uniquely at every point z in the cut z-plane. It is only necessary to show that it agrees with some previous definition, such as the one given in $(5)_a$ for $|z-1|<2$. Hence taking the path p as a circle of radius $2-0$, center at 1, the integrand is developable in ascending powers of $(z-1)/(z+1)$ giving

$$P_\nu^m(z) = \frac{(z^2-1)^{\frac{m}{2}}}{2^\nu \Gamma(\nu+1)} \sum_{s=0}^{\infty} (z-1)^s \frac{\Gamma(s+m+\mu+1)}{\Gamma(s+1)} \cdot \frac{1}{2\pi i}\int_p \frac{(z'+1)^\nu \, dz'}{(z'-1)^{s+m+1}}$$

Since

$$\frac{1}{2\pi i}\int_p \frac{(z'+1)^\nu}{(z'-1)^{s+m+1}}\, dz' = \frac{1}{\Gamma(s+m+1)}\Big[D_z (z+1)^\nu\Big]_{z=1} = \frac{2^{\nu-m-s}\,\Gamma(\nu+1)}{\Gamma(s+m+1)\,\Gamma(1+\nu-m-s)}$$

this becomes the definition $(5)_a$ so that $(74)_a$ is proven. Similarly $(74)_a$ is verified when $|z|>1$. From it one derives

75) $$Q_\nu^m(z) = \frac{(z^2-1)^{\frac{m}{2}}(-1)^m\,\Gamma(\nu+m+1)}{2^{\nu+1}\,\Gamma(\nu+1)} \int_{-1}^{1} \frac{(1-t^2)^\nu \, dt}{(z-t)^{\nu+m+1}} \quad \text{if } \Re(\nu+1)>0$$

the plane being cut from $-\infty$ to $+1$ as for (74).

For Neumann's formula

76) $Q_n(z) = \frac{1}{2}\int_{-1}^{1} P_n(t)\,\frac{dt}{z-t}$ where n is a positive integer, the

z-plane need only be cut (to render this valid for all values of z) along the real axis of z from -1 to 1.

This is also the cut in Heine's formula

77) $$\frac{1}{z'-z} = \sum_{n=0}^{\infty} (2n+1) P_n(z)\, Q_n(z')$$

which is valid if z is inside the ellipse with foci at ±1 which passes through z'.

This is analogous to the expansion of Frobenius

78) $$\frac{1}{2(y-x)} \log\frac{(x+1)(y-1)}{(x-1)(y+1)} = \sum_{n=0}^{\infty} (2n+1)\, Q_n(x)\, Q_n(y)$$

If $R(z)>0$ the path p in (74)a may be taken as a circle with center at z and radius $|\sqrt{z^2-1}|-0$. The substitution $z'-z = \sqrt{z^2-1}\, e^{i\alpha'}$ converts (74)a into Laplace's integral (71).

8 Addition-theorem for $P_\nu(z)$ and $Q_n(z)$.

It is necessary to show that the integral on the left side of eq(72) is equal to $P_\nu(z)$ where z is given by$(72)_a$. Since $m=0$ Schläfli's integral$(74)_a$ becomes

79) $$P_\nu(z) = \frac{1}{2^{\nu+1}\pi i}\int_p \frac{(z'^2-1)^\nu dz'}{(z'-z)^{\nu+1}}$$ where the z-plane need

only be cut from $-\infty$ to -1. Let the path p be considered as any one of the infinitely many circular paths each of which encloses the points z and $+1$ but not -1. Changing the variable from z' to t by a homographic substitution converts the circular path p into a circular path in the t-plane. It is necessary for the proof here sketched that this be the unit circle with center at the origin so that on the new path $t = e^{i\alpha'}$ where α' is a real variable ranging from $-\pi$ to π as the circle is described. A substitution which is of the type $z' = z + A\left(\frac{t-B}{1-Ct}\right)$ where $|B|<1$ ad $|C|<1$, represents the interior of the circle $|t|=1$ upon the interior of a circle in the plane of the variable z' and the fixed point z is inside the z'-circle (or path p).

This is evident from the fact that when $z' = z$, $t = B$ and $|t| = |B| < 1$. Also when $t = \frac{1}{C}$ $|t| = \frac{1}{|C|} > 1$ so the exterior of the circle $|t| = 1$ represents the exterior of the circle which is its transform. Hence the interiors of the two circles correspond. The path p will be a permissible path for the integral (79) if the constants A, B, C are such that the transform of $z' = +1$ is inside, and that of $z' = -1$ outside, the circle $|t| = 1$.

These conditions are all satisfied by the substitution

80)$_a$ $$z' - z = A\left(\frac{t - i\tan\frac{\omega_2}{2}}{1 - i\tan\frac{\omega_2}{2}t}\right)$$

where

80)$_b$ $$A = 2i\cos^2\frac{\omega_1}{2}\cos^2\frac{\omega_2}{2}\left(1 + \tan\frac{\omega_1}{2}\tan\frac{\omega_2}{2}e^{i\phi}\right)\left(\tan\frac{\omega_2}{2} - \tan\frac{\omega_1}{2}e^{-i\phi}\right)$$

where ϕ is real. To prove this we note that the hypothesis upon which eq (72) rests is that $R(z_1) > 0$ and $R(z_2) > 0$ so that $|\tan\frac{\omega_1}{2}|$ and $|\tan\frac{\omega_2}{2}|$ are both less than unity. No further hypothesis is necessary (Whitaker & Watson's proof is limited by the further assumption that $R(z) > 0$ where z is defined by (72)$_a$.).

Since $|\tan\frac{\omega_2}{2}| < 1$ eqn (80)$_a$ shows that the interior of the circle $|t| = 1$ represents the interior of a circular path in the plane of the variable z', which path encloses the fixed point z. It remains to be

proven that this path encloses the point $z'=+1$ but not the point $z'=-1$. To show this we write the eq $(72)_a$ in the equivalent form

$$81)_a \quad \cos w = \cos w_1 \cos w_2 + \sin w_1 \sin w_2 \cos\phi$$

From this we find

$$81)_a \quad z-1 = \cos w - 1 = -2\cos^2\frac{w_1}{2}\cos^2\frac{w_2}{2}\left(\tan\frac{w_2}{2} - \tan\frac{w_1}{2}e^{i\phi}\right)\left(\tan\frac{w_2}{2} - \tan\frac{w_1}{2}e^{-i\phi}\right)$$

$$81)_b \quad z+1 = \cos w + 1 = 2\cos^2\frac{w_1}{2}\cos^2\frac{w_2}{2}\left(1 + \tan\frac{w_1}{2}\tan\frac{w_2}{2}e^{i\phi}\right)\left(1 + \tan\frac{w_1}{2}\tan\frac{w_2}{2}e^{-i\phi}\right)$$

Using these in $80)_a$ and $(80)_b$ gives

$$82)_a \quad z'-1 = 2i\cos^2\frac{w_1}{2}\left[\frac{t - i\tan\frac{w_1}{2}}{1 - i\tan\frac{w_1}{2}\,t}\right]\left(\tan\frac{w_2}{2} - \tan\frac{w_1}{2}e^{-i\phi}\right)$$

$$82)_b \quad z'+1 = 2\cos^2\frac{w_1}{2}\left[\frac{1 - i\tan\frac{w_1}{2}\,t}{1 - i\tan\frac{w_1}{2}\,t}\right]\left(1 + \tan\frac{w_1}{2}\tan\frac{w_2}{2}e^{i\phi}\right)$$

The point $t = i\tan\frac{w_1}{2}$ inside the circle $|t|=1$ is by $(82)_a$ the transform of the point $z'=1$, but by $(82)_b$ the transform of the point $z'=+1$ is $t = \frac{1}{i\tan\frac{w_1}{2}}$ outside the circle $|t|=1$.

Consequently the circular path $t = e^{i\alpha'}$ is the transform of a circular z' path for which Schläfli's integral (79) is valid.

The further details of this transformation require the use of the two identities

$$83)_a \quad z_1 + \sqrt{z_1^2-1}\cos(\phi-\alpha') \equiv \cos^2\frac{\omega_1}{2}\left(1 - i\tan\frac{\omega_1}{2} e^{i(\alpha'-\phi)}\right)\left(1 - i\tan\frac{\omega_1}{2} e^{-i(\alpha'-\phi)}\right)$$

$$83)_b \quad z_2 + \sqrt{z_2^2-1}\cos\alpha' \equiv \cos^2\frac{\omega_2}{2}\left(1 - i\tan\frac{\omega_2}{2} e^{i\alpha'}\right)\left(1 - i\tan\frac{\omega_2}{2} e^{-i\alpha'}\right)$$

By use of these and $(80)_a$ when $t = e^{i\alpha'}$ we find

$$84)_a \quad \frac{dz'}{z'-z} = \frac{i\,d\alpha'}{z_2 + \sqrt{z_2^2-1}\cos\alpha'}$$

and from $82)_a$ and $82)_b$ using $(83)_a$ we find

$$84)_b \quad z'^2-1 = 4i\cos^2\frac{\omega_1}{2}\, e^{i\alpha'} \frac{[z_1+\sqrt{z_1^2-1}\cos(\phi-\alpha')]\,[\text{[illegible]}\tan\frac{\omega_1}{2} e^{i\phi}]}{[1 - i e^{i\alpha'}\tan\frac{\omega_2}{2}]^{2}}\left[\tan\frac{\omega_2}{2} - \tan\frac{\omega_1}{2} e^{-i\phi}\right]$$

Dividing this by $z'-z$ of $(80)_a$ gives

$$84)_c \quad \frac{z'^2-1}{z'-z} = \frac{2\left[z_1 + \sqrt{z_1^2-1}\cos(\phi-\alpha')\right]}{z_2 + \sqrt{z_2^2-1}\cos\alpha'}$$

Using $(84)_a$ and $(84)_c$ in (79) gives

$$85) \quad P_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\left[z_1 + \sqrt{z_1^2-1}\cos(\phi-\alpha')\right]^\nu}{\left[z_2 + \sqrt{z_2^2-1}\cos\alpha'\right]^{\nu+1}}\,d\alpha'$$

Eqn (85) and (72) establish the addition theorem

$$86)\quad P_\nu(z_1 z_2 - \sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\phi) = 2\sum_{m=0}^{\infty} \epsilon_m (-1)^m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(z_1) P_\nu^{-m}(z_2)\cos m\phi$$

where $R(z_1)>0$ and $R(z_2)>0$ This may be written

$$86)'\quad T_\nu(z_1 z_2 + \sqrt{1-z_1^2}\sqrt{1-z_2^2}\cos\phi) = 2\sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} T_\nu^m(z_1) T_\nu^{-m}(z_2)\cos m\phi$$

An expansion analogous to this when n is a positive integer is that of Heine-Neumann is given by Whitaker

$$87)\quad Q_n(z_1 z_2 - \sqrt{z_1^2-1}\sqrt{z_2^2-1}\cos\phi) = 2\sum_{m=0}^{\infty} \epsilon_m (-1)^m \frac{\Gamma(n-m+1)}{\Gamma(n+m+1)} Q_n^m(z_1) P_n^m(z_2)\cos m\phi$$

where $\left|\frac{z_2-1}{z_2+1}\right| < \left|\frac{z_1-1}{z_1+1}\right|$ and ϕ real.

The conditions under which the expansion (86) hold are precisely those under which each function $P_\nu^m(z_1)$ and $P_\nu^m(z_2)$ is expansible in terms of $Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}}(\zeta_1)$ and $Q_{m-\frac{1}{2}}^{\nu+\frac{1}{2}}(\zeta_2)$ by Whipple's eq (62)$_e$ where $\zeta_i = \frac{z_i}{\sqrt{z_i^2-1}}$, for replacing μ by m and then ν by $\mu-\frac{1}{2}$, eq (62)$_e$ gives

$$P_{\mu-\frac{1}{2}}^m(z_1) = \sqrt{\frac{2}{\pi}}(-1)^m\,\Gamma(m+\tfrac{1}{2}-\mu)\left(\frac{\zeta_1^2-1}{\pi}\right)^{\frac{1}{4}} Q_{m-\frac{1}{2}}^{\mu}(\zeta_1)$$

Hence after replacing the ζ_1 by z we obtain the

following form of the addition-theorem

$$88)_a \quad P_{\mu-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right) = \frac{4(z_1^2-1)^{\frac{1}{4}}(z_2^2-1)^{\frac{1}{4}}}{\pi^2 \cos\mu\pi} \sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma(m+\frac{1}{2}-\mu)}{\Gamma(m+\frac{1}{2}+\mu)} Q^{\mu}_{m-\frac{1}{2}}(z_1)\, Q^{\mu}_{m-\frac{1}{2}}(z_2) \cos m\phi$$

valid, like (86), when $R(z_1) > 0$ and $R(z_2) > 0$.

If in addition to this $R\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right) > 0$, then by $(62)_e$

$$P_{\mu-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right) = \sqrt{\frac{2}{\pi}} \frac{(z_1^2-1)^{\frac{1}{4}}(z_2^2-1)^{\frac{1}{4}}[z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi]^{-\frac{1}{4}}}{\cos\mu\pi\ \Gamma(\mu+\frac{1}{2})} \cdot Q^{\mu}_{-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi}}\right)$$

so that $(88)_a$ may be written

$$88)_b \quad [z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi]^{-\frac{1}{4}} \cdot Q^{\mu}_{-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi}}\right) =$$

$$= \left(\frac{2}{\pi}\right)^{3/2} \Gamma(\mu+\tfrac{1}{2}) \sum_{m=0}^{\infty} \epsilon_m \frac{\Gamma(m+\frac{1}{2}-\mu)}{\Gamma(m+\frac{1}{2}+\mu)} Q^{\mu}_{m-\frac{1}{2}}(z_1)\, Q^{\mu}_{m-\frac{1}{2}}(z_2) \cos m\phi$$

A case of some importance in potential problems is $\mu = 0$

$$88)_c \quad [z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi]^{-\frac{1}{4}} Q_{-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{z_1^2+z_2^2-2z_1z_2\cos\phi-\sin^2\phi}}\right) =$$

$$= \frac{\pi}{\sqrt{2}} (z_1^2-1)^{-\frac{1}{4}} (z_1^2-1)^{-\frac{1}{4}} P_{-\frac{1}{2}}\left(\frac{z_1 z_2 - \cos\phi}{\sqrt{(z_1^2-1)(z_2^2-1)}}\right) =$$

$$= \frac{2\sqrt{2}}{\pi} \sum_{m=0}^{\infty} \epsilon_m Q_{m-\frac{1}{2}}(z_1)\, Q_{m-\frac{1}{2}}(z_2) \cos m\phi$$

Several addition theorems for $Q_{m-\frac{1}{2}}$ are given in section X where it is shown that there is an infinite number of such expansions.

9. Large values of the parameters.

(a). $|\nu| \to \infty$ (μ fixed).

If z is any point in the cut z-plane except on either side of a cut where z is a real less than -1 then $|t|<1$ where $t = \dfrac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}$ as in (59)c. The point t is on one of the two circular sheets of fig 2 but not on the circumference of either. Eqn (61)a is

$$Q_\nu^\mu(z) = 2^\mu \frac{\sqrt{\pi}\cos\mu\pi\,(z^2-1)^{\frac{\mu}{2}}}{(z+\sqrt{z^2-1})^{\mu+\nu+1}}\, f_\nu^\mu(t)$$

where

$$f_\nu^\mu(t) = \frac{1}{\Gamma(\mu+\frac{1}{2})}\sum_{s=0}^{\infty} t^s \frac{\Gamma(s+\mu+\frac{1}{2})}{\Gamma(s+1)}\,\frac{\Gamma(\nu+s+\mu+1)}{\Gamma(\nu+s+\frac{3}{2})}$$

When $\nu \to \infty$ and $|\arg \nu| < \pi - \epsilon$

$\dfrac{\Gamma(\nu+s+\mu+1)}{\nu^{\mu-\frac{1}{2}}\Gamma(\nu+s+\frac{3}{2})} \to 1$ so that every term of finite order s

in the series defining $\dfrac{f_\nu^\mu(t)}{\nu^{\mu-\frac{1}{2}}}$ approaches the term in the development

$$\frac{1}{\Gamma(\mu+\frac{1}{2})}\sum_{s=0}^{\infty} t^s \frac{\Gamma(s+\mu+\frac{1}{2})}{\Gamma(s+1)} = \frac{1}{(1-t)^{\mu+\frac{1}{2}}} = \left(\frac{z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right)^{\mu+\frac{1}{2}}$$

It may be shown that, in the asymptotic sense,

$$f_\nu^\mu(t) \simeq \left(\frac{z+\sqrt{z^2-1}}{2\sqrt{z^2-1}}\right)^{\mu+\frac{1}{2}} \nu^{\mu-\frac{1}{2}} \quad ---- |\arg\nu| < \pi-\epsilon$$

Hence if $\nu \to \infty$ and $|\arg \nu| < \pi$

89)$_a$ $$Q^{\mu}_{\nu}(z) \approx \sqrt{\frac{\pi}{2}}\frac{\cos\mu\pi}{(z^2-1)^{\frac{1}{4}}}\frac{\nu^{\mu-\frac{1}{2}}}{(z+\sqrt{z^2-1})^{\nu+\frac{1}{2}}} = \sqrt{\frac{\pi}{2}}\cdot\frac{\cos\mu\pi}{(z^2-1)^{\frac{1}{4}}}\,\nu^{\mu-\frac{1}{2}}\,e^{i(\nu+\frac{1}{2})\omega}$$

where $z = \cos\omega$, $\sqrt{z^2-1} = e^{-\frac{i\pi}{2}}\sin\omega$ as in (64)a, b, c

In this expression it is assumed that $\nu \to \infty$ in any direction except that of the negative real axis. To obtain P^{μ}_{ν} from (61)$_c$ we require $f^{\mu}_{-\nu-1}(t)$ for the same restrictions on ν as in (89)$_a$. To find an asymptotic expression (the first term only) for $P^{\mu}_{\nu}(z)$ we consider that $\nu \to \infty$ in any direction except the (positive and negative) real directions. Writing $f^{\mu}_{-\nu-1}(t)$ in the form

$$f^{\mu}_{-\nu-1}(t) = -\frac{\cos\nu\pi}{\sin(\nu-\mu)\pi}\frac{1}{\Gamma(\mu+\frac{1}{2})}\sum_{s=0}^{\infty} t^s\,\frac{\Gamma(s+\mu+\frac{1}{2})}{\Gamma(s+1)}\,\frac{\Gamma(\nu+\frac{1}{2}-s)}{\Gamma(\nu+1-\mu-s)}$$

$$\left.\begin{aligned}&\frac{1}{\nu^{\mu-\frac{1}{2}}}\,\frac{\Gamma(\nu+\frac{1}{2}-s)}{\Gamma(\nu+1-\mu-s)} \to 1\\ &-\frac{\cos\nu\pi}{\sin(\nu-\mu)\pi} \to e^{\mp i(\mu-\frac{1}{2})\pi}\end{aligned}\right\}\quad \nu_2 \to \binom{+\infty}{-\infty} \quad \text{where } \nu = \nu_1 + i\nu_2$$

so that

89)$_b$ $$\left.\begin{aligned}Q^{\mu}_{-\nu-1}(z) &\approx \sqrt{\frac{\pi}{2}}\frac{\cos\mu\pi}{(z^2-1)^{\frac{1}{4}}}\,e^{\mp i(\mu-\frac{1}{2})\pi}\,\nu^{\mu-\frac{1}{2}}(z+\sqrt{z^2-1})^{\nu+\frac{1}{2}}\\ &= \sqrt{\frac{\pi}{2}}\frac{\cos\mu\pi}{(z^2-1)^{1/4}}\,e^{\mp i(\mu-\frac{1}{2})\pi}\,\nu^{\mu-\frac{1}{2}}\,e^{-i(\nu+\frac{1}{2})\omega}\end{aligned}\right\} \quad \nu_2 \to \binom{+\infty}{-\infty}$$

This gives

$$90)\quad P_\nu^\mu(z) \approx \frac{\nu^{\mu-\frac{1}{2}}}{\sqrt{2\pi}\,(z^2-1)^{\frac{1}{4}}}\left[-e^{\pm i(\mu-\frac{1}{2})\pi}\,e^{i(\nu+\frac{1}{2})w} + e^{-i(\nu+\frac{1}{2})w}\right] \text{ when } \nu_2 \to \binom{+\infty}{-\infty}$$

Since $R\ \pm i(\nu+\frac{1}{2})w = \mp[\nu_2\alpha + (\nu_1+\frac{1}{2})\beta]$ it is evident that only one of the two exponential terms is ∞, this being the only one to retain.

If we let $\beta \to 0$ while $-\pi < \alpha < 0$ this corresponds to $z = x + i0$ and $(z^2-1)^{\frac{1}{4}} = e^{+\frac{i\pi}{4}}\sqrt{1-x^2} = e^{\frac{i\pi}{4}}\sqrt{|\sin\alpha|}$

so that (90) becomes

$$e^{-i\frac{\mu\pi}{2}} P_\nu^\mu(x+i0) \equiv T_\nu^\mu(x) \approx \frac{e^{\pm i\mu\pi}}{\sqrt{2\pi|\sin\alpha|}}\left(\nu e^{\mp\frac{i\pi}{2}}\right)^{\mu-\frac{1}{2}} e^{\pm i(\nu+\frac{1}{2})\alpha} \quad \tfrac{\nu}{2} \to \binom{+\infty}{-\infty}$$

Or letting $\alpha = -\theta$ so that $x = \cos\theta$, $0 < \theta < \pi$

$$91)\quad T_\nu^\mu(\cos\theta) \approx \frac{e^{\pm i\mu\pi}}{\sqrt{2\pi\sin\theta}}\left(\nu e^{\mp\frac{i\pi}{2}}\right)^{\mu-\frac{1}{2}} e^{\mp i(\nu+\frac{1}{2})\theta} \quad \text{when } \nu_2 \to \binom{+\infty}{-\infty}$$

From this by $13)_a$ we find

$$92)\quad Q_\nu^\mu(\cos\theta) \approx \mp\frac{i\pi\cos\mu\pi}{2\sqrt{2\pi\sin\theta}}\left(\nu e^{\mp\frac{i\pi}{2}}\right)^{\mu-\frac{1}{2}} e^{\mp i(\nu+\frac{1}{2})\theta} \quad \text{when } \nu_2 \to \binom{+\infty}{-\infty}$$

From (91) we find for the "spindle function" when $0 < \theta < \pi$ and ν_2 real

93)$_a$ $$T^{m}_{-\frac{1}{2}+i\nu_2}(\cos\theta) \approx \frac{(-1)^m}{\sqrt{2\pi \sin\theta}} |\nu_2|^{m-\frac{1}{2}} e^{|\nu_2|\theta} \quad \text{when } \nu_2 \to \pm\infty$$

93)$_b$ $$T^{m}_{-\frac{1}{2}+i\nu_2}(-\cos\theta) \equiv T^{m}_{-\frac{1}{2}+i\nu_2}(\cos(\pi-\theta)) \approx \frac{(-1)^m}{\sqrt{2\pi \sin\theta}} |\nu_2|^{m-\frac{1}{2}} e^{|\nu_2|(\pi-\theta)} \quad \text{when } \nu_2 \to \pm\infty$$

Barnes has established an asymptotic formula for $T^{\mu}_{\nu}(\cos\theta)$ which has been shown by Watson to be valid for $|\arg\nu| < \pi-\epsilon$. It is (Hobson p 304) in the present notation (see 26)$_d$).

94)$_a$ $$T^{\mu}_{\nu}(\cos\theta) \approx \frac{2\,\nu^{\mu-\frac{1}{2}}}{\sqrt{2\pi \sin\theta}} \left\{ \cos\left[(\nu+\tfrac{1}{2})\theta - \tfrac{\pi}{2}(\mu+\tfrac{1}{2})\right] + O\left(\tfrac{1}{\nu}\right) \right\} \quad \text{for } |\arg\nu| < \pi-\epsilon$$

so that by 13)$_a$

94)$_b$ $$q^{\mu}_{\nu}(\cos\theta) \approx -\cos\mu\pi \sqrt{\frac{\pi}{2\sin\theta}} \cdot \nu^{\mu-\frac{1}{2}} \left\{ \sin\left[(\nu+\tfrac{1}{2})\theta + \tfrac{\pi}{2}(\mu-\tfrac{1}{2})\right] + O\left(\tfrac{1}{\nu}\right) \right\} \quad \text{for } |\arg\nu| < \pi-\epsilon$$

These are the same as 91) and (92) when ν is complex.

(b) $|\mu| \to \infty$ $\quad \nu$ fixed

When $R(z) > 0$, eqn (45), treated similarly, gives

$$95)_a \quad \Gamma(\nu-\mu+1)\, P_\nu^\mu(z) \approx \left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} \mu^\nu$$

$$95)_b \quad \Gamma(\nu-\mu+1)\, \mathsf{P}_\nu^\mu(z) \approx \left(\frac{1-z}{1+z}\right)^{\frac{\mu}{2}} \mu^\nu$$

$0 \leq |\arg \mu| < \pi - \epsilon$

ϵ positive small

Eq (5)$_b$ and (12)$_a$ show these are also valid inside the circle $|z-1| = 2$. Since P_ν^μ is a single-valued function of μ, these give

$$\Gamma(\nu+\mu+1)\, P_\nu^{-\mu}(z) \approx \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} (\mu e^{-i\pi})^\nu \quad \text{if} \quad \pi \geq \arg\mu > \epsilon$$

$$\approx \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} (\mu e^{i\pi})^\nu \quad \text{if} \quad -\pi \leq \arg\mu < -\epsilon,$$

that is

$$96)_a \quad \Gamma(\nu+\mu+1)\, P_\nu^{-\mu}(z) \approx \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} (\mu e^{\mp i\pi})^\nu$$

$$96)_b \quad \Gamma(\nu+\mu+1)\, \mathsf{P}_\nu^{-\mu}(z) \approx \left(\frac{1-z}{1+z}\right)^{-\frac{\mu}{2}} (\mu e^{\mp i\pi})^\nu$$

$R(z) > 0$ or $|z-1| > 2$

$\mu_2 \to \begin{matrix} +\infty \\ -\infty \end{matrix}$

where $\mu = \mu_1 + i\mu_2$

Hence by (96)$_b$ and (13)$_a$ we find if $-1 < x < 1$

$$97)_a \quad \Gamma(\nu-\mu+1)\, \mathsf{Q}_\nu^\mu(x) \approx \pm \frac{i\pi}{2} \mu^\nu \left[\left(\frac{1-x}{1+x}\right)^{\frac{\mu}{2}} - \left(\frac{1-x}{1+x}\right)^{-\frac{\mu}{2}} \frac{e^{\mp i(\nu+\mu)\pi}}{2}\right] \quad \mu_2 \to \begin{pmatrix} +\infty \\ -\infty \end{pmatrix}$$

Also by (96)$_a$ and (6)$_a$

$$\Gamma(\nu-\mu+1)\, Q_\nu^\mu(z) \approx \pm \frac{i\pi}{2} \mu^\nu \left[\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} - \left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} e^{\mp i\nu\pi}\right] \quad \mu_2 \to \begin{pmatrix} +\infty \\ -\infty \end{pmatrix}$$

In this equation only one term survives so that for $R(z)>0$

$$97)_b \quad \Gamma(\nu-\mu+1)\, Q_\nu^\mu(z) \approx \mp \frac{i\pi}{2}\left(\frac{z-1}{z+1}\right)^{-\frac{\mu}{2}} \left(\mu\, e^{\mp i\pi}\right)^\nu \cdots \mu_2 \to \left(\begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}\right), \quad \mu_1 > 0 \text{ or } +\infty$$

$$\approx \pm \frac{i\pi}{2}\left(\frac{z-1}{z+1}\right)^{\frac{\mu}{2}} \mu^\nu \cdots \mu_2 \to \begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}, \quad \mu_1 < 0, \text{ or } -\infty$$

For the case $R(z)<0$ we find from (47)

$$\Gamma(\nu-\mu+1)\, Q_\nu^\mu(z) \approx \mp \frac{i\pi}{2}\left(\frac{1-z}{z-1}\right)^{\nu} \mu^\nu \left[\left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} - \left(\frac{z+1}{z-1}\right)^{-\frac{\mu}{2}} e^{\mp i\nu\pi}\right] \cdots \mu_2 \to \left(\begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}\right)$$

Only one term survives here. Also by (11) $\frac{1-z}{z-1} = e^{\mp i\pi}$ if $y \lessgtr 0$

Hence if $x<0$ and $y>0$ (in region I of fig 1)

$$98)_a \quad \Gamma(\nu-\mu+1)\, Q_\nu^\mu(z) \approx \pm \frac{i\pi\left(\mu\, e^{-i\pi}\right)^\nu}{2} \left(\frac{z+1}{z-1}\right)^{-\frac{\mu}{2}} e^{\mp i\nu\pi} \cdots \mu_2 \to \left(\begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}\right), \quad \mu_1 > 0$$

$$\approx \mp \frac{i\pi}{2}\left(\mu\, e^{-i\pi}\right)^\nu \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \cdots \mu_2 \to \left(\begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}\right), \quad \mu_1 < 0$$

But if $x<0$ and $y<0$ (in region IV of fig 1)

$$98)_b \quad \Gamma(\nu-\mu+1)\, Q_\nu^\mu(z) \approx \pm \frac{i\pi}{2}\left(\mu\, e^{i\pi}\right)^\nu \left(\frac{z+1}{z-1}\right)^{-\frac{\mu}{2}} e^{\mp i\nu\pi} \cdots \mu_2 \to \left(\begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}\right), \quad \mu_1 > 0$$

$$\approx \mp \frac{i\pi}{2}\left(\mu\, e^{i\pi}\right)^\nu \left(\frac{z+1}{z-1}\right)^{\frac{\mu}{2}} \cdots \mu_2 \to \begin{smallmatrix}+\infty\\-\infty\end{smallmatrix}; \quad \mu_1 < 0$$

Using $(98)_a$ in $(6)_b$ shows that $(95)_a$ also holds when $x<0$, $y>0$ if $\mu_1>0$

VII Heun's Functions

The "generalized" hypergeometric function is a solution of a differential equation of higher order than the second but with three singular points, but the generalization which arises in some physical problems is Heun's function which satisfies a homogeneous linear equation of second order having four singular points, all regular. Since the equation preserves this character under any homographic transformation of the independent variable, three of the singular points could be brought to $z = 0, 1,$ and ∞, the fourth at $z = a$ being then any point in the z-plane. It must then be of the form, as shown by Fuchs

1)
$$\begin{cases} \dfrac{d^2\bar{y}}{dz^2} + \bar{P}(z)\dfrac{d\bar{y}}{dz} + \bar{Q}(z)\dfrac{d\bar{y}}{dz} = 0 \\[2ex] \bar{P} = \dfrac{1-\bar{\alpha}_0-\bar{\beta}_0}{z} + \dfrac{1-\bar{\alpha}_1-\bar{\beta}_1}{z-1} + \dfrac{1-\bar{\alpha}_2-\bar{\beta}_2}{z-a} \\[2ex] Q = \dfrac{1}{\psi(z)}\left[\bar{\alpha}\bar{\beta}z + \bar{b} + \bar{\alpha}_0\bar{\beta}_0\dfrac{\psi'(0)}{z} + \dfrac{\bar{\alpha}_1\bar{\beta}_1\psi'(1)}{z-1} + \dfrac{\bar{\alpha}_2\bar{\beta}_2\psi'(a)}{z-a}\right] \\[2ex] \text{where} \\ \psi(z) \equiv z(z-1)(z-a) \end{cases}$$

The exponents are
$$\begin{array}{llll} \bar\alpha_0, \bar\beta_0 & \text{for } z=0 & = z_0 \\ \bar\alpha_1, \bar\beta_1 & \text{for } z=1 & = z_1 \\ \bar\alpha_2, \bar\beta_2 & \text{for } z=a & = z_2 \\ \alpha, \beta & \text{for } z=\infty & = z_4 \end{array}$$
They are not all assignable since they must satisfy $\bar\alpha + \bar\beta + \sum_{n=0}^{2}(\bar\alpha_n + \bar\beta_n) = 2$.
The constant b is arbitrary.
The equation (1) becomes the normal form when $\bar\alpha_0 = \bar\alpha_1 = \bar\alpha_2 = 0$. If not in that form it becomes so by the change of the dependent variable from $\bar y$ to y by

2) $$\bar y = z^{\bar\alpha_0}(z-1)^{\bar\alpha_1}(z-a)^{\bar\alpha_2}\, y$$

where

3)
$$\frac{d}{dz}\left[z^{\gamma}(z-1)^{1+\alpha+\beta-\gamma-\delta}(z-a)^{\delta}\frac{dy}{dz}\right] + z^{\gamma-1}(z-1)^{\alpha+\beta-\gamma-\delta}(z-a)^{\delta-1}\left[\alpha\beta z + b\right]y = 0$$
or
$$\frac{d^2y}{dz^2} + P\frac{dy}{dz} + \frac{\alpha\beta z + b}{z(z-1)(z-a)}\,y = 0$$
$$P = \frac{\gamma}{z} + \frac{1+\alpha+\beta-\gamma-\delta}{z-1} + \frac{\delta}{z-a} = \frac{d}{dz}\log z^{\gamma}(z-1)^{1+\alpha+\beta-\gamma-\delta}\cdot(z-a)^{\delta}$$
$$\alpha \equiv \bar\alpha + \sum_{n=0}^{2}\bar\alpha_n, \quad \beta \equiv \bar\beta + \sum_{n=0}^{2}\bar\alpha_n, \quad \gamma \equiv 1+\bar\alpha_0-\bar\beta_0, \quad \delta = 1+\bar\alpha_2-\bar\beta_2$$

The constant b is given by

$$4)\quad b = \bar{b} + \bar{\alpha}_0\left[\bar{\beta}_0(a+1) + (\bar{\beta}_1 - 1)a + \bar{\beta}_2 - 1\right]$$

$$+ \bar{\alpha}_1\left[(\bar{\beta}_0 - 1)a + \bar{\beta}_1(a-1)\right] + \bar{\alpha}_2\left[\bar{\beta}_0 - 1 - \bar{\beta}_2(a-1)\right]$$

The exponents in (3) are

zero and $1-\gamma$ at $z=0$

zero and $\gamma+\delta-\alpha-\beta$ at $z=1$

zero and $1-\delta$ at $z=a$

α and β at $z=\infty$

In the particular case where $\gamma=\delta=\frac{1}{2}$, $\alpha=\pm\frac{m}{2}$, $\beta=\mp\left(\frac{m+}{2}\right.$ equation (3) becomes the (algebraic form of) Lamé-equation

$$5)_a\quad y'' + \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-a}\right)y' + \frac{b - \frac{1}{4}m(m+1)}{z(z-1)(z-a)}\,y = 0$$

If m is a non-negative integer this is the Lamé-Hermite equation. Replacing m by $m-\frac{1}{2}$ gives the Lamé-Wangerin equation

$$5)_b\quad y'' + \frac{1}{2}\left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{z-a}\right)y' + \frac{b + \frac{1}{4}\left(\frac{1}{4} - m^2\right)}{z(z-1)(z-a)}\,y = 0$$

where m is an integer. $\left(\alpha = \frac{1}{4} + \frac{m}{2},\ \beta = \frac{1}{4} - \frac{m}{2}\right)$

The solution of (3) which is regular in the neighborhood of $z=0$ and belongs to the exponent zero is Heun's function defined by the series

$$6)_a \quad y_1(z) = F(a, b; \alpha, \beta, \gamma, \delta; z) \equiv 1 - \frac{b}{\gamma a} z + \sum_{s=2}^{\infty} c_s z^s$$

The solution which belongs to the exponent $1-\gamma$ is

$$6)_b \begin{cases} y_2(z) = z^{1-\gamma} F(a, b_2; 1+\alpha-\gamma, 1+\beta-\gamma, 2-\gamma, \delta; z) \\ \text{where} \\ b_2 = b - (1-\gamma)[\delta + a(1+\alpha+\beta-\gamma-\delta)] \end{cases}$$

These and their derivatives satisfy

$$6)_c \quad y_1(z) y_2'(z) - y_1'(z) y_2(z) = \frac{1-\gamma}{z^{\gamma} (1-z)^{1+\alpha+\beta-\gamma-\delta} \cdot (1-\frac{z}{a})^{\delta}}$$

The coefficients c_s are determined by the difference-equation

$$7) \quad (s+2)(s+1+\gamma) a \, c_{s+2} = \left\{(s+1)^2(a+1) + (s+1)[\gamma+\delta-1+(\alpha+\beta-\delta)a] - b\right\} c_{s+1}$$

$$- (s+\alpha)(s+\beta) c_s$$

with the initial conditions

$$7)' \quad c_0 = 1, \quad c_1 = -\frac{b}{\gamma a} \quad \text{and } c_s = 0 \text{ if } s < 0$$

The definition fails if $\gamma = 0$ for y_1, and if $\gamma = 2$ for y_2, and if $\gamma = 1$, $y_1(z) \equiv y_2(z)$. These two solutions become identical except for a constant factor if γ is any integer for by (7) we find that if n is any non-negative integer

$$\lim_{\gamma \to -n} \frac{y_1}{\Gamma(\gamma)} = \frac{c_{n+2}}{\Gamma(n+2)} \Big[y_2(z) \Big]_{\gamma = -n}$$

The procedure for constructing functions analogous to the g-functions in I could be followed. For example in the case $\gamma = 1$ (where y_1 and y_2 each exist but are identical) another solution could be found in the form

$$g(z) = \lim_{\gamma \to 1} \left[\frac{y_1(z) - y_2(z)}{\gamma - 1} \right]$$

$$= F(a, b; \alpha, \beta, 1, \delta; z) \log z$$

$$+ \left(D_\alpha + D_\beta + 2 D_\gamma - [\delta + (\alpha + \beta - \delta) a] D_b \right) F(a, b; \alpha, \beta, \gamma, \delta; z)_{\gamma = 1}$$

or

8) $$g(z) = F(a, b; \alpha, \beta, 1, \delta; z) \log z + \sum_1^\infty a_s z^s$$

Such cases involving logarithms arise when the indicial equation has multiple roots. They are not considered here.

The series (6)a and (6)b converge inside a circle with center at the origin whose radius is the distance from the origin to the nearer of the two singular points a and 1.

In the case $|a| > 1$ we find

If $\Re(\gamma+\delta-\alpha-\beta) > 0$, $y_1(1)$ and $y_2(1)$ converge

If $\Re(\gamma+\delta-\alpha-\beta) \leq 0$ both diverge (if they are infinite series).

This is found from the recurrence relation (7), for if u_s is the n^{th} term of the series (6)a $\frac{u_{s+1}}{u_s} \to 1$

but

$$\log s - (s+1)\frac{u_{s+1}}{u_s}\log(s+1) = -1 + (\gamma+\delta-\alpha-\beta)\log s$$

+ terms which vanish with $\frac{1}{s}$.

From the difference equation (7) one finds the following cases in which Heun's function degenerates into the hypergeometric function. (or g-function if $\gamma, \alpha, \gamma+\delta = -n$)

By confluence of the singular points a and 0 when $b=0$

9)$_a$ $F(0,0;\alpha,\beta,\gamma,\delta;z) = F(\alpha,\beta,\gamma+\delta;z)$ if $\gamma+\delta \neq -n$

Confluence of a and ∞ gives

9)$_b$ $\begin{cases} F(a, ae; \alpha,\beta,\gamma,\delta; z)_{a\to\infty} = F(\mu+\sqrt{\mu^2+e}, \mu-\sqrt{\mu^2+e}, \gamma, z) \quad \text{if } \gamma\neq -n \\ \text{where } \mu \equiv \frac{\alpha+\beta-\delta}{2} \end{cases}$

9)$_c$ $\begin{cases} F(1, b; \alpha,\beta,\gamma,\delta;z) = (1-z)^{\frac{\gamma-\alpha-\beta}{2}+\mu} F\left(\frac{\gamma+\alpha-\beta}{2}+\mu, \frac{\gamma-\alpha+\beta}{2}+\mu, \gamma; z\right) \\ \text{where } \mu = \pm\sqrt{\left(\frac{\gamma-\alpha-\beta}{2}\right)^2 - \alpha\beta - b} \quad \text{if } \gamma\neq -n \end{cases}$

If $b = -\alpha\beta$ this becomes $F(\alpha, \beta, \gamma; z)$

It is evident from (7) that Heun's function is unaltered by interchange of α and β. Also, if z is inside the circle of convergence $|z| < |a|$ or $|z| < 1$, Heun's function converges without placing restriction upon $\alpha, \beta, \gamma, \delta$ or b and is therefore an integral function of these parameters. In particular when $|a| > 1$ and $\gamma + \delta - \alpha - \beta = 0$, $z = 1$, Heun's function defines an integral function of b.

When α or β is a negative integer (and γ is not) the hypergeometric function becomes a polynomial in z. The analogous character of y_1 is obtained by suitable choice of b which is the Bernoulli constant whose characteristic values make the solutions of (3) satisfy certain boundary conditions.

If $\alpha = -m$ where m is a given non-negative integer the equation (7) determines all the coefficients c_s beginning with $c_0 = 1$, $c_1 = -\frac{b}{\gamma a}$ up to c_{m+1} inclusive. For $s = m$ eq (7) becomes

$$10)_a \quad (m+2)(m+1+\gamma)a\,c_{m+2} = \{(m+1)^2(a+1) + (m+1)[\gamma+\delta-1+(\beta-m-\delta)a] - b\}\,c_{m+1}$$

where c_{m+1} is a known polynomial in b of degree $m+1$

whose coefficients depend upon a, β, γ, δ and $\alpha = m$
Hence if b is taken as one of the $m+1$ roots
of the algebraic equation in b

10)$_b$ $C_{m+1}(b) = 0$ (parameters $a, b; -m, \beta, \gamma, \delta;$)

then $c_s = 0$ for $s > m$ by 10)$_a$ and (7) and the Heun's function y_1 becomes a polynomial in z of the m^{th} degree. There are $(m+1)$ polynomial solutions of (3) if the roots of (10)$_b$ are distinct.

In the Lame-Hermite equation $(5)_a$ $\alpha\beta = -\left(\frac{m}{2}\right)\left(\frac{m+1}{2}\right)$
and we may take either $\alpha = -\frac{m}{2}$, $\beta = \frac{m+1}{2}$
or $\alpha = +\frac{m}{2}$, $\beta = -\left(\frac{m+1}{2}\right)$ so that whether m be even or odd, the equation has polynomial solutions.

There are, when m is even,

11)$_a$
$$\begin{cases} y_n^m(z) = F\left(a, b_n; -\frac{m}{2}, \frac{m+1}{2}, \frac{1}{2}, \frac{1}{2}; z\right) \text{ for } n = 1, 2, 3, \cdots \frac{m}{2}+1 \\ \text{where } b_n \text{ is the } n^{th} \text{ root of} \\ C_{\frac{m}{2}+1}(b) = 0 \quad \text{with parameters } \left(a, b; -\frac{m}{2}, \frac{m+1}{2}, \frac{1}{2}, \frac{1}{2};\right) \end{cases}$$

If m is odd they are

11)$_b$
$$\begin{cases} y_n^m(z) = F\left(a, b_n; \frac{m}{2}, -\left(\frac{m+1}{2}\right), \frac{1}{2}, \frac{1}{2}; z\right) \text{ for } n = 1, 2, \cdots \frac{m+1}{2}+1 \\ \text{where } b_n \text{ is the } n^{th} \text{ root of} \\ C_{\frac{m+1}{2}+1}(b) = 0 \text{ with parameters } \left(a, b; \frac{m}{2}, -\left(\frac{m+1}{2}\right), \frac{1}{2}, \frac{1}{2};\right) \end{cases}$$

These solutions are all of the type $y_1(z)$ in $(6)_a$. Corresponding to each eigen-value b there is another solution of type $y_2(z)$ in $(6)_b$ which will never be a finite polynomial.

To obtain solutions of the Lamé-Wangerin equation $(5)_b$ in finite form, the quadratic transformation obtained in eq $(27)_a$ below may be used. This corresponds to Landen's transformations of elliptic functions, for if in equ $(5)_b$ $z = sn^2(\alpha, K)$ the periodic form of that equation becomes, letting $a = \frac{1}{K^2}$

$$12)_a \quad \frac{d^2 y}{d\alpha^2} + \left[\left(\tfrac{1}{4} - m^2\right) K^2 sn^2\alpha + 4K^2 b\right] y = 0$$

which becomes

$$12)_b \quad \frac{d^2 y}{d\alpha_1^2} + \left[\left(\tfrac{1}{4} - m^2\right) \frac{K_1^4\, sn^2\alpha_1\, cn^2\alpha_1}{dn^2\alpha_1} + 4(1-K_1')^2 b\right] y = 0$$

by Landen's transformation

$$12)_c \quad K^2 sn^2(\alpha, K) = (1-K_1')^2 \frac{sn^2(\alpha_1, K_1)\, cn^2(\alpha_1, K_1)}{dn^2(\alpha, K_1)} \quad \text{where } K = \frac{1-K_1'}{1+K_1'} \text{ and } K_1' \equiv \sqrt{1-K_1^2}$$

Equ $(27)_a$ gives

$$12)_d \quad F\left(\tfrac{1}{K^2}, b; \tfrac{1}{4}+\tfrac{m}{2}, \tfrac{1}{4}-\tfrac{m}{2}, \tfrac{1}{2}, \tfrac{1}{2}; z\right) =$$

$$= (1-K_1^2 z_1)^{\frac{1}{4}-\frac{m}{2}} (1-z_1)^{\frac{1}{2}} \bar{F}\left(\tfrac{1}{K_1^2}, \bar{b}; 1-m, 1, \tfrac{1}{2}, 1-m; z_1\right)$$

where

12)$_e$ $\bar{b} = b\,\dfrac{1-\sqrt{1-K_1^2}}{1+\sqrt{1-K_1^2}} - \dfrac{1}{4}\left(\dfrac{1}{K_1^2} + m + \dfrac{1}{2}\right)$.

and

12)$_f$ $2z_1 = 1 + Kz - \sqrt{(1-z)(1-K^2 z)}$ so that when $z = sn^2\alpha$, $z_1 = sn^2\alpha_1$

The function in second member of (12)$_d$ permit solutions which are polynomials in z_1 if m is a positive integer, but not in the case $m=0$.

To obtain transformation formulae for the Heun's function there are twenty-four homographic substitutions which transform eq(3) into the form (1) in y and the new variable z', the new singular points being $z'=0, 1, a', \infty$ where a' denotes the fourth singular point and not the transform of a in general. The normal form (3) is recovered by a change of the dependent variable as in (2). In this way forty-eight solutions of (3) are obtained in terms of Heun's functions with different parameters and arguments: Any three whose domains have a part in common must be connected by a homogeneous linear relation. A few of the relations thus obtained are given here.

The six transformations in which z' vanishes with z are

$$z' = z,\ \frac{z}{a},\ \frac{z}{z-1},\ \frac{z}{z-a},\ \frac{(a-1)z}{a(z-1)},\ \frac{(1-a)z}{z-a}$$

The six in which z' vanishes at $z=1$ are

$$z' = 1-z,\ \frac{z-1}{a-1},\ \frac{z-1}{z},\ \frac{z-1}{z-a},\ \frac{a(z-1)}{z-a},\ \frac{a(z-1)}{(a-1)z}$$

The six in which z' vanishes at $z=a$ are

$$z' = \frac{a-z}{a},\ \frac{a-z}{a-1},\ \frac{a-z}{1-z},\ \frac{a-z}{a(1-z)},\ \frac{z-a}{z},\ \frac{z-a}{(1-a)z}$$

The last six in which z' vanishes at $z=\infty$ are

$$z' = \frac{1}{z},\ \frac{a}{z},\ \frac{1}{1-z},\ \frac{1-a}{1-z},\ \frac{a}{a-z},\ \frac{a-1}{a-z}$$

For the analytic continuation of the function $y_1(z)$ the plane should be cut from 1 to ∞ and from a to ∞ to render y_1, $(1-z)^\alpha$ and $(1-\frac{z}{a})^\alpha$ single valued. For the function y_2 an additional cut is made from zero to $-\infty$ to make z^α single-valued.

The first substitution $z' = \frac{z}{a}$ gives

13) $$F(a, b; \alpha, \beta, \gamma, \delta; z) = F\left(\frac{1}{a}, \frac{b}{a}; \alpha, \beta, \gamma, 1+\alpha+\beta-\gamma-\delta; \frac{z}{a}\right)$$

The substitution $z' = z/(z-1)$ gives one analogue of Euler's Theorem

14) $$F(a, b; \alpha, \beta, \gamma, \delta; z) =$$
$$= (1-z)^{-\alpha} F\left(\frac{a}{a-1}, -\frac{(b+a\alpha\gamma)}{a-1}; \alpha, \gamma+\delta-\beta, \gamma, \delta; \frac{z}{z-1}\right)$$

where α and β may be interchanged.

Another is the continuation

15) $F(a, b; \alpha, \beta, \gamma, \delta; z) =$

$$= \left(1-\frac{z}{a}\right)^{-\alpha} F\left(\frac{1}{1-a}, -\frac{(b+\alpha\gamma)}{1-a}; \alpha, 1+\alpha-\delta, \gamma, 1+\alpha+\beta-\gamma-\delta; \frac{z}{z-a}\right)$$

where α and β may be interchanged.

By combinations of these, the following identities come

$16)_a$ $F(a, b; \alpha, \beta, \gamma, \delta; z) =$

$$= (1-z)^{\gamma+\delta-\alpha-\beta} F(a, b-a\gamma(\gamma+\delta-\alpha-\beta); \gamma+\delta-\alpha, \gamma+\delta-\beta, \gamma, \delta; z)$$

$16)_b$ $F(a, b; \alpha, \beta, \gamma, \delta; z) =$

$$= \left(1-\frac{z}{a}\right)^{1-\delta} \cdot F(a, b-\gamma(1-\delta); \alpha+1-\delta, \beta+1-\delta, \gamma, 2-\delta; z)$$

$16)_c$ $F(a, b; \alpha, \beta, \gamma, \delta; z) =$

$$= (1-z)^{\gamma+\delta-\alpha-\beta} \cdot \left(1-\frac{z}{a}\right)^{1-\delta} \cdot F(a, b'; 1+\gamma-\alpha, 1+\gamma-\beta, \gamma, 2-\delta; z)$$

where

$b' = b - \gamma(\gamma+\delta-\alpha-\beta)a - \gamma(1-\delta)$

To examine the domain of existence of such functions as in the second member of (14) one must look not only at the argument but also at the first parameter which is the corresponding singular point. Thus if $a = a_1 + i a_2$ and $z = x + iy$ equation (14) is valid when $x < \frac{1}{2}$ if $a_1 > \frac{1}{2}$ but when $a_1 < \frac{1}{2}$ the point z

must be inside the circle which passes through a and whose center is on the negative real axis at $z = -\frac{a_1^2+a_2^2}{1-2a_1}$

Similarly eq(15) is valid for the case $|a-1|<1$ when z on the same side of the perpendicular bisector of the line $\overline{0a}$ as the origin. But when $|a-1|>1$ eq(15) is valid inside a circle which encloses the origin and passes through (1,0) its equation being

$$\left(x+\frac{a_1}{a_1^2+a_2^2-2a_1}\right)^2+\left(y+\frac{a_2}{a_1^2+a_2^2-2a_1}\right)^2=\frac{(a_1^2+a_2^2)\left[(a_1-1)^2+a_2^2\right]}{(a_1^2+a_2^2-2a_1)^2}$$

In these and the following equations, when the point z is not in the region common to the domain of existence of both members, the equation gives the analytic continuation of one member.

The substitution $z' = a(1-z)/(a-z)$ gives two solutions of (3)

17)$_a$
$$\begin{cases} y_3(z) = \left(\dfrac{1-\frac{z}{a}}{1-\frac{1}{a}}\right)^{-\alpha} F\left(a, b_3; \alpha, 1+\alpha-\delta, 1+\alpha+\beta-\gamma-\delta, 1+\alpha-\beta; \dfrac{1-z}{1-\frac{z}{a}}\right) \\ \text{where} \\ b_3 = b-\alpha(1+\alpha-\gamma-\delta) \end{cases}$$

17)$_b$
$$\begin{cases} y_4(z) = \\ = (1-z)^{\gamma+\delta-\alpha-\beta}\left(\dfrac{1-\frac{z}{a}}{1-\frac{1}{a}}\right)^{\beta-\gamma-\delta} F\left(a, b_4; \gamma+\delta-\beta, \gamma+1-\beta, 1+\gamma+\delta-\alpha-\beta, 1+\alpha-\beta; \dfrac{1-z}{1-\frac{z}{a}}\right) \\ \text{where} \\ b_4 = b-(1-\beta)(\gamma+\delta-\beta)-a\gamma(\gamma+\delta-\alpha-\beta) \end{cases}$$

17)$_c$
$$y_3 y_4' - y_3' y_4 = \frac{(\alpha+\beta-\gamma-\delta)}{z^{\gamma}(1-z)^{1+\alpha+\beta-\gamma-\delta}}\left(\frac{1-\frac{1}{a}}{1-\frac{z}{a}}\right)^{\delta}.$$

To obtain a transformation analogous to Gauss's for the hypergeometric function consider the case where $|a|>1$ and write 6)$_a$ 6)$_b$ (17)$_a$ (17)$_b$ respectively as

$y_1(z) = F_1(z)$ $\quad y_2(z) = z^{1-\gamma} F_2(z)$, $\quad y_3(z) = \left(\frac{1-\frac{z}{a}}{1-\frac{1}{a}}\right)^{-\alpha} F_3\left(\frac{1-z}{1-\frac{z}{a}}\right)$ and

$y_4(z) = (1-z)^{\gamma+\delta-\alpha-\beta} \left(\frac{1-\frac{z}{a}}{1-\frac{1}{a}}\right)^{\beta-\gamma-\delta} F_4\left(\frac{1-z}{1-\frac{z}{a}}\right)$

If we assume that $1-\gamma$ and $\gamma+\delta-\alpha-\beta$ are not integers and their real parts are positive, then

$F_1(1)$, $F_2(1)$ $F_3(1)$, $F_4(1)$ converge and $y_4(1) = y_2(0) = 0$.

Since the domain of y_1 & y_2 has a region common to that of y_3 and y_4 we find

18) $$F_2(1) = \frac{(1-\gamma)(1-\frac{1}{a})^{\gamma-\beta}}{\gamma+\delta-\alpha-\beta} \cdot F_4(1)$$

and

19)$_a$ $y_1(z) \equiv F_1(z) =$

$$= \left(\frac{1-\frac{z}{a}}{1-\frac{1}{a}}\right)^{-\alpha} \left\{ F_1(1) F_3\left(\frac{1-z}{1-z/a}\right) + \left[\left(1-\frac{1}{a}\right)^{-\alpha} - F_1(1) F_3(1)\right] \left(\frac{1-z}{1-\frac{z}{a}}\right)^{\gamma+\delta-\alpha-\beta} \cdot \frac{F_4\left(\frac{1-z}{1-z/a}\right)}{F_4(1)} \right\}$$

19)$_b$ $y_2(z) \equiv z^{1-\gamma} F_2(z) =$

$$= \left(\frac{1-z/a}{1-1/a}\right)^{-\alpha} \left\{ F_2(1) F_3\left(\frac{1-z}{1-z/a}\right) - \frac{(1-\gamma)(1-\frac{1}{a})^{\gamma-\beta}}{\gamma+\delta-\alpha-\beta} F_3(1) \left(\frac{1-z}{1-z/a}\right)^{\gamma+\delta-\alpha-\beta} \cdot F_4\left(\frac{1-z}{1-z/a}\right) \right\}$$

where α and β may be interchanged.

The substitution $z' = 1-z$ in (3) leads to a different expression for y_3 and y_4

20)$_a$ $y_3(z) = F(1-a, -b-\alpha\beta; \alpha, \beta, 1+\alpha+\beta-\gamma-\delta, \delta; 1-z)$

20)$_b$ $$\begin{cases} y_4(z) = (1-z)^{\gamma+\delta-\alpha-\beta} F(1-a, b_4'; \gamma+\delta-\alpha, \gamma+\delta-\beta, 1+\gamma+\delta-\alpha-\beta, \delta; 1-z) \\ b_4' = -b-\alpha\beta-(\gamma+\delta-\alpha-\beta)(\gamma+\delta-a\gamma) \end{cases}$$

Other continuations for y_3 and y_4 are obtainable by applying (14) to the F-functions in (17). The results is of the form ( ) $F(\frac{a}{a-1}, -; -, -, -, -, \frac{a(z-1)}{z})$

The substitution $z' = \frac{a}{z}$, $y = z^{\alpha} Y$ changes the normal equation (3) into another with parameters

$a' = a$, $b' = b - \alpha[1+\alpha-\gamma-\delta+a(\delta-\beta)]$

$\alpha' = \alpha$, $\beta' = 1+\alpha-\gamma$, $\gamma' = 1+\alpha-\beta$, $\delta' = 1+\alpha+\beta-\gamma-\delta$

Hence we find two new solutions of (3), $y_5(z)$ and $y_6(z)$

21)$_a$ $$\begin{cases} y_5(z) = z^{-\alpha} F(a, b_5; \alpha, 1+\alpha-\gamma, 1+\alpha-\beta, 1+\alpha+\beta-\gamma-\delta; \frac{a}{z}) \\ \text{where} \\ b_5 = b - \alpha[1+\alpha-\gamma-\delta+a(\delta-\beta)] \end{cases}$$

The other solution $y_6(z)$ for the same domain is found to be the result of interchanging α and β in y_5

Continuing this by (14) gives

$$21)_b \begin{cases} y_5(z) = (z-a)^{-\alpha} F\left(\frac{a}{a-1}, b_5'; \alpha, 1+\alpha-\delta, 1+\alpha-\beta, 1+\alpha+\beta-\gamma-\delta; \frac{a}{a-z}\right) \\ b_5' = \dfrac{-b+\alpha[1+\alpha-\gamma-\delta-a(1+\alpha-\delta)]}{a-1} \end{cases}$$

Applying (15) to $(21)_a$ gives

$$21)_b \begin{cases} y_5(z) = (z-1)^{-\alpha} F\left(\frac{1}{1-a}, b_5''; \alpha, \gamma+\delta-\beta, 1+\alpha-\beta, \delta; \frac{1}{z-1}\right) \\ b_5'' = \dfrac{-b+\alpha[\beta-\gamma-\delta+a(\delta-\beta)]}{1-a} \end{cases}$$

In all forms of y_5 we may interchange α and β and, if $\alpha \neq \beta$, get a distinct solution y_6.

The substitution $z' = \frac{a-z}{a-1}$ leaves (3) in the normal form with parameters $a' = \frac{a}{a-1}$, $b' = -\left(\frac{b+a\alpha\beta}{a-1}\right)$, $\alpha' = \alpha$, $\beta' = \beta$, $\gamma' = \delta$, $\delta' = \gamma$.

Two new solutions of (3) thus found are

$$22)_a \begin{cases} y_7(z) = F\left(\frac{a}{a-1}, b_7; \alpha, \beta, \delta, \gamma; \frac{a-z}{a-1}\right) \\ b_7 = -\dfrac{b+a\alpha\beta}{a-1} \end{cases}$$

$$22)_b \begin{cases} y_8(z) = \left(\frac{a-z}{a-1}\right)^{1-\delta} F\left(\frac{a}{a-1}, b_8; 1+\alpha-\delta, 1+\beta-\delta, 2-\delta, \gamma; \frac{a-z}{a-1}\right) \\ b_8 = \dfrac{1}{a-1}\left\{-b-a\alpha\beta+(1-\delta)[\gamma-a(1+\alpha+\beta-\delta)]\right\} \end{cases}$$

The continuation of these by use of (14) is

$$23)_a \quad \begin{cases} y_7(z) = \left(\frac{z-1}{a-1}\right)^{-\alpha} F(a, b_7'; \alpha, \gamma+\delta-\beta, \delta, \gamma; \frac{z-a}{z-1}) \\ b_7' = b + a\alpha(\beta-\delta) \end{cases}$$

$$23)_b \quad \begin{cases} y_8(z) = \left(\frac{z-1}{a-1}\right)^{-\alpha} \cdot \left(\frac{a-z}{z-1}\right)^{1-\delta} F(a, b_8'; 1+\alpha-\delta, 1-\beta+\gamma, 2-\delta, \gamma; \frac{z-a}{z-1}) \\ b_8' = b - \gamma(1-\delta) + a(\beta-1)(1+\alpha-\delta) \end{cases}$$

The continuation got by using (15) on (22) is

$$24)_a \quad \begin{cases} y_7(z) = \left(\frac{z}{a}\right)^{-\alpha} F(1-a, b_7''; \alpha, 1+\alpha-\gamma, \delta, 1+\alpha+\beta-\gamma-\delta; \frac{z-a}{z}) \\ b_7'' = -\left[b + \alpha\delta + a\alpha(\beta-\delta)\right] \end{cases}$$

$$24)_b \quad \begin{cases} y_8(z) = \left(\frac{a}{a-1}\right)^{1-\delta} \left(\frac{a-z}{z}\right)^{1-\delta} \left(\frac{a}{z}\right)^{\alpha} F(1-a, b_8''; 1+\alpha-\delta, 2+\alpha-\gamma-\delta, 2-\delta, 1+\alpha+\beta-\gamma-\delta; \frac{z-a}{z}) \\ b_8'' = -b - \alpha\gamma + (1+\alpha-\delta)\left[\gamma+\delta-2 + a(1-\beta)\right] \end{cases}$$

In the special case where $\gamma = \alpha + \beta$ and $\delta = \frac{1}{2}$ the variable z in (3) may be replaced by z_1 where z_1 is that root of the quadratic equation

25)$_a$ $$\kappa z = \frac{z_1(1-z_1)}{a_1 - z_1} = \frac{\kappa_1^2 z_1(1-z_1)}{1-\kappa_1^2 z_1}$$ which vanishes with z,

where $a \equiv \frac{1}{\kappa^2}$ and $a_1 = \left(\frac{a^{\frac{1}{2}}+a^{-\frac{1}{2}}}{2}\right)^2 \equiv \frac{1}{\kappa_1^2}$ the two κs being connected by the relation between moduli in Landen's transformation that is, $\kappa_1 = \frac{2\sqrt{\kappa}}{1+\kappa}$ so that $\kappa = \frac{1-\kappa_1'}{1+\kappa_1'}$ where $\kappa_1' = \sqrt{1-\kappa_1^2}$

The solution of (25)$_a$ for z_1 is

25)$_b$ $$z_1 = \frac{1}{2}\left[1 + \kappa z - \sqrt{(1-z)(1-\kappa^2 z)}\right]$$ where the radical denotes the branch which is $+1$ when $z \to 0$ so that z and z_1 vanish together.

Letting $y = (1-\kappa_1^2 z_1)^{\alpha}\, Y(z_1)$ the equation (3) transforms into a similar normal form in Y and z_1 with parameters

26) $$a_1 \equiv \frac{1}{\kappa_1^2},\ b_1 \equiv b\frac{(1-\sqrt{1-\kappa_1^2})}{1+\sqrt{1-\kappa_1^2}} - \alpha(\alpha+\beta);\ \alpha_1 \equiv 2\alpha,\ \beta_1 \equiv \gamma_1 = \alpha+\beta,\ \delta_1 \equiv 1+\alpha-\beta.$$

We thus obtain

27)$_a$ $$y_1(z) \equiv F\left(\tfrac{1}{\kappa^2}, b; \alpha, \beta, \alpha+\beta, \tfrac{1}{2}; z\right) =$$

$$= (1-\kappa_1^2 z_1)^{\alpha} F\left(\tfrac{1}{\kappa_1^2}, b_1; 2\alpha, \alpha+\beta, \alpha+\beta, 1+\alpha-\beta; z_1\right)$$

$$= (1-z_1)^{1-\alpha-\beta}\cdot(1-\kappa_1^2 z_1)^{\beta} F\left(\tfrac{1}{\kappa_1^2}, b_1'; 1-\alpha+\beta, 1, \alpha+\beta, 1-\alpha+\beta; z_1\right)$$ by (16)$_c$

where $$b_1' = b\frac{1-\sqrt{1-\kappa_1^2}}{1+\sqrt{1-\kappa_1^2}} - (\alpha+\beta)\left[\alpha + \frac{1-\alpha-\beta}{\kappa_1^2}\right]$$

and

$$y_2(z) \equiv z^{1-\alpha-\beta} F\left(\frac{1}{K^2}, b + \frac{(\alpha+\beta-1)(1+K^2)}{2K^2}; 1-\alpha, 1-\beta, 2-\alpha-\beta, \frac{1}{2}; z\right) \qquad 27)_b$$

$$= (1-K_1^2 z_1)^{\alpha} \left[(1+K_1')^2 z_1\right]^{1-\alpha-\beta} F\left(\frac{1}{K_1^2}, b_2; 1+\alpha-\beta, 1, 2-\alpha-\beta, 1+\alpha-\beta; z_1\right)$$

where

$$b_2 = b\,\frac{1-\sqrt{1-K_1^2}}{1+\sqrt{1-K_1^2}} - \alpha(\alpha+\beta) - (1-\alpha-\beta)\left(1+\alpha-\beta+\frac{\alpha+\beta}{K_1^2}\right).$$

VIII Generalizations of Fourier's Integral

1. General formulation.

The function $f(x)$ to be represented as an integral need not itself be real although x is considered a real variable. The values of $f(x)$ may be assigned arbitrarily subject to the following restrictions

1)a $f(x)$ is continuous except at a finite number of points, none of which are limit points, and all are such that $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges.

1)b When $x \to \pm\infty$, $f(x) \to 0$, and in such a manner that $\lim_{x \to \pm\infty} e^{\kappa|x|} f(x) = 0$ if $\kappa < \delta$ where δ is a given positive constant. In case $f(x) \equiv 0$ when $|x| > x_0$ then $\delta = \infty$. In case $f(x) \sim C_1 e^{-\delta_1 x}$ when $x \to +\infty$ and $f(x) \sim C_2 e^{\delta_2 x}$ when $x \to -\infty$, take δ equal to the smaller of the positive constants δ_1 and δ_2.

For such a function Fourier's integral may be put in the form, wherever $f(x)$ is not infinite

$$2)_a \quad \frac{1}{2}\left[f(x+0)+f(x-0)\right]=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} e^{-\mu(x+iy)}\,d\mu\int_{-\infty}^{\infty} f(x_1)\,e^{\mu(x_1+iy)}\,dx_1$$

or

$$2)_b \quad \frac{1}{2}\left[f(x+0)+f(x-0)\right]=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} e^{\mu(x+iy)}\,d\mu\int_{-\infty}^{\infty} f(x_1)\,e^{-\mu(x_1+iy)}\,dx_1$$

where $-\delta<\mu_1<\delta$, the integration being taken up the line $\mu=\mu_1$ of plane of the complex variable $\mu\equiv\mu_1+i\mu_2$. The constant y which cancels out is put in to indicate more general possibility

Let $z\equiv x+iy$ and let the analogue of $e^{-\mu z}$ be $E_1^{\mu}(z)\equiv e^{-\mu z}\cdot p_1^{\mu}(z)$ and let $e^{\mu z}$ be analogous to $E_2^{\mu}(z)\equiv e^{\mu z}\, p_2^{\mu}(z)$, these being solutions of the differential equation

$$3) \quad E''(z)+\left[q(z)-\mu^2\right]E(z)=0 \qquad (E'(z)=D_z E)$$

so that

$$3)_a \quad p_1''^{\mu}(z)-2\mu\, p_1'^{\mu}(z)+q\, p_1^{\mu}(z)=0$$

and

$$3)_b \quad p_2''^{\mu}(z)+2\mu\, p_2'^{\mu}(z)+q\, p_2^{\mu}(z)=0$$

The given function $q(z)$ is assumed to be an analytic function of z in the strip of the z-plane

$$4)_a \quad -b<y<b \quad \text{and} \quad -\infty<x<\infty \quad \text{which vanishes}$$

when $x \to \pm\infty$. (If $q(z)$ should vanish like $e^{-2c|x|}$ when $|x| \to \infty$, then the solutions of (3) might have a singularity if $\mu = -c$) In the first part of the discussion it is assumed that μ is a point in the half-plane

4)$_b$ $\quad 0 < \mu_1 < \infty$ and $-\infty < \mu_2 < \infty$

while z is in the strip (4)$_a$ of the z-plane.

For this range of μ and z we make the following assumptions as to the character of the functions $p_1^\mu(z)$ and $p_2^\mu(z)$ which are solutions of (3)$_a$ and (3)$_b$ respectively

A) $\quad p_1^\mu(z)$ is an analytic function of the two complex variables z and μ.

B) $\quad p_1^\mu(z) \to 1$ when $|\mu| \to \infty$

C) $\quad p_1^\mu(z) \to 1$ when $x \to +\infty$

D) $\quad p_1^\mu(z) \to \dfrac{\Omega(0,\mu,\mu)}{2\mu}$ when $x \to -\infty$

And

A)$'$ $\quad p_2^\mu(z)$ is an analytic function of z and of μ.

B)$'$ $\quad p_2^\mu(z) \to 1$ when $|\mu_2| \to \infty$

C)$'$ $\quad p_2^\mu(z) \to 1$ when $x \to -\infty$

D)$'$ $\quad p_2^\mu(z) \to \dfrac{\Omega(0,\mu,\mu)}{2\mu}$ when $x \to +\infty$

The function $\Omega(z,\mu,\lambda)$ is defined as an analytic function of z, μ, and λ ($\equiv \lambda_1 + i\lambda_2$) when when μ and λ are in the half plane(4)$_e$, by

$$5)_a \quad \Omega(z,\mu,\lambda) \equiv E_1^{\mu}(z) E_2^{\prime\lambda}(z) - E_1^{\prime\mu}(z) E_2^{\lambda}(z) =$$
$$= e^{(\lambda-\mu)z}\left\{(\lambda+\mu) p_1^{\mu}(z) p_2^{\lambda}(z) + p_1^{\mu}(z) p_2^{\prime\lambda}(z) - p_1^{\prime\mu}(z) p_2^{\lambda}(z)\right\}$$

so that

$$5)_b \quad D_z \Omega(z,\mu,\lambda) = (\lambda^2 - \mu^2) E_1^{\mu}(z) E_2^{\lambda}(z)$$

When $\lambda = \mu$, Ω becomes independent of z which is indicated by writing

$$5)_c \quad \Omega(z,\mu,\mu) = \Omega(0,\mu,\mu) = 2\mu\, p_1^{\mu}(z) p_2^{\mu}(z) + p_1^{\mu}(z) p_2^{\prime\mu}(z) - p_1^{\prime\mu}(z) p_2^{\mu}(z)$$

which by assumptions (A) and (A') is an analytic function of μ in the half plane (4)$_e$.

The assumed conditions C, D, C' and D' require that $p_1^{\prime\mu}(z)$ and $p_2^{\prime\mu}(z)$ vanish when $x \to \pm\infty$; these four conditions then are compatible, for if we let $x \to +\infty$ or $-\infty$ in eq (5)$_c$ we get D' or D. Also by B or B' we find that

$$5)_d \quad \frac{\Omega(0,\mu,\mu)}{2\mu} \to 1 \text{ when } |\mu| \to \infty$$

The points of the μ-plane which are roots of the equation $\Omega(0,\mu,\mu)=0$, are values of μ for which the functions $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ become linearly dependent. The assumptions B_1 and B' require that for large positive values of μ_1, $E_1^{\mu}(z) \sim e^{-\mu z}$ and $E_2^{\mu}(z) \sim e^{\mu z}$ which are linearly independent. Consequently the equation has at most a finite number of roots in the half plane $(4)_b$. If c_0 is the real part of the root whose real part is greatest, then

$5)_e$ $\quad \Omega(0,\mu,\mu) \neq 0$ when $c_0 < \mu_1 < \infty$, $\quad -\infty < \mu_2 < \infty$

In particular cases c_0 may be zero or negative.

Any function of x of class f may be developed in the Fourier's integrals of form $(2)_a$ or $(2)_b$ but it is only "developable" with respect to $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ when its positive constant δ exceeds c_0

The condition of developability is

$5)_b$ $\quad c_0 < \delta$

The generalizations of Fourier's integrals in the form $(2)_a$ or $(2)_b$ which are to be obtained may be written

$$6)_a \quad \frac{1}{2}\left[f(x+0)+f(x-0)\right]=\frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0+i\infty}\frac{2\mu\, E_1^{\mu}(x+iy)}{\Omega(0,\mu,\mu)}\,d\mu\int_{-\infty}^{\infty}f(x_1)\,E_2^{\mu}(x_1+iy)\,dx_1$$

and

$$6)_b \quad \frac{1}{2}\left[f(x+0)+f(x-0)\right]=\frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0-i\infty}\frac{2\mu\, E_2^{\mu}(x+iy)}{\Omega(0,\mu,\mu)}\,d\mu\int_{-\infty}^{\infty}f(x_1)\,E_1^{\mu}(x_1+iy)\,dx_1$$

The constant y cancels out of the exponential parts only, since $E_1^{\mu}(x+iy)E_2^{\mu}(x_1+iy)=e^{\mu(x_1-x)}p_1^{\mu}(x+iy)\,p_2^{\mu}(x_1+iy)$

In the limiting case where $q(z)\equiv 0$ in eq (3) the functions $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ degenerate into $e^{-\mu z}$ and $e^{\mu z}$ respectively, ($p_1^{\mu}(z)\equiv p_2^{\mu}(z)\equiv 1$) and $\Omega(0,\mu,\mu)\equiv 2\mu$. Hence eq (6) reduce to eq (2).

In these and the following integrals the variables of integration are μ_2 and x while μ_1 and y being constants may occasionally be suppressed in the notation.

To derive (6) it is first necessary to consider some of the properties of the functions of μ defined by the x_1 integrals.

A function of μ may be called of class F if it has the following character

$7)_a$ $F(\mu)$ is analytic in the strip $0 < \mu_1 < \delta$ and $-\infty < \mu_2 < \infty$

$7)_b$ $\displaystyle\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} F(\mu)\, d\mu$ converges

$7)_c$ $\displaystyle\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{F(\mu)\, d\mu}{\mu - \lambda}$ converges absolutely if $\lambda_1 \neq \mu_1$

It is also developable if its constant δ satisfies $(5)_b$

It must first be shown that the two transforms of $f(x)$

$8)_a$ $$F_1(\mu) \equiv \int_{-\infty}^{\infty} f(x) e^{-\mu x} p_1^{\mu}(x+iy)\, dx$$

and

$8)_b$ $$F_2(\mu) \equiv \int_{-\infty}^{\infty} f(x) e^{\mu x} p_2^{\mu}(x+iy)\, dx$$

are of class F with the same δ, and that these two integrals converge absolutely and uniformly as to μ_2 in any finite interval.

Consider $F_2(\mu)$.

If μ_1 has any positive value less than δ we may interpolate a value a_2 where $\mu_1 < a_2 < \delta$ and it follows from $(1)_a$ and $(1)_\ell$ that $\int_{-\infty}^{\infty} |f(x)| e^{a_2 x} dx$ converges. Hence if ϵ is any preassigned positive constant, arbitrarilly small, a positive constant x_0 depending upon ϵ may be taken so large that $\int_{x_0}^{\infty} |f(x)| e^{a_2 x} dx < \epsilon$.

Now

$$F_2(\mu) = \int_{-\infty}^{x_0} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx + \int_{x_0}^{\infty} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx$$

By assumptions A_1' and C_1' a positive constant M_1 depending on x_0 but not upon μ may be found such that $|p_2^{\mu}(x+iy)| < M_1$ when $-\infty \leq x \leq x_0$

Also by D_1' a positive constant, M_2, depending on x_0 but not upon μ, exists such that

$$|p_2^{\mu}(x+iy)| < M_2 \left|\frac{\Omega(0\,\mu,\mu)}{2\mu}\right| \quad \text{when} \quad x_0 \leq x < \infty$$

Hence

$$\left|\int_{-\infty}^{x_0} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx\right| < M_1 \int_{-\infty}^{x_0} f(x) e^{-\mu_1 x} dx < M_1 \int_{-\infty}^{\infty} |f(x)| e^{a_1 x} dx$$

and

$$\left|\int_{x_0}^{\infty} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx\right| < M_2 \left|\frac{\Omega(0,\mu\mu)}{2\mu}\right| \int_{x_0}^{\infty} |f(x)| e^{\mu_1 x} dx$$

$$< \epsilon M_2 \left|\frac{\Omega(0\,\mu\,\mu)}{2\mu}\right|$$

This shows that the integral defining $F_2(\mu)$ is absolutely convergent when μ is any finite point in the strip $0<\mu_1<\delta$. Moreover by $A)$ and $A)'$ and $(5)_d$ there is a positive constant m depending upon η such that $|\Omega(0,\mu,\mu)/2\mu| < m$ when $-\eta<\mu_2<\eta$. Consequently the integral defining $F_2(\mu)$ converges uniformly as to μ_2 in any finite interval. The proof for $F_1(\mu)$ is similar so that $F_1(\mu)$ and $F_2(\mu)$ are analytic in the strip $0<\mu_1<\delta$.

Also

$$\operatorname{limit}_{\mu_2\to\pm\infty} \left| \int_{x_0}^{\infty} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx \right| < \epsilon M_2 \lim_{\mu_2\to\pm\infty} \left| \frac{\Omega(0,\mu,\mu)}{2\mu} \right| = \epsilon M_2$$

Since ϵ is arbitrary this limit is zero. Hence

$$\lim_{\mu_2\to\pm\infty} F_2(\mu) = \lim_{|\mu_2|\to\infty} \int_{-\infty}^{x_0} f(x) e^{\mu x} p_2^{\mu}(x+iy)\,dx$$

$$= \lim_{|\mu_2|\to\infty} \int_{-\infty}^{x_0} [f(x) e^{\mu_1 x} p_2^{\mu}(x+iy)](\cos\mu_2 x + i\sin\mu_2 x)\,dx$$

$= 0$, by the theorem of Riemann-Lebesgue, since $\int_{-\infty}^{x_0} |f(x) e^{\mu_1 x} p_2^{\mu}(x+iy)|\,dx$ is convergent.

Hence

$F_1(\mu)\to 0$ and $F_2(\mu)\to 0$ when $\mu_2\to\pm\infty$

This result would follow if, $f(x)e^{\mu_1 x}p_2^{\mu}(x+iy)$,

while being absolutely integrable, merely vanishes when $x \to \pm\infty$, without making use of its postulated exponential vanishing. On account of the latter it is found that integrals like $(7)_e$ converge and those like $(7)_e$ converge absolutely.

Consider first the case where $f(x)$ is everywhere continuous. Integration by parts gives

$$\mu F_2(\mu) \equiv h(\mu) = -\int_{-\infty}^{\infty} e^{\mu_1 x}\left[f(x)\, p_2'^{\mu}(x+iy) + f'(x)\, p_2^{\mu}(x+iy)\right](\cos\mu_2 x + i \sin\mu_2 x)\,dx$$

so that

$h(\mu) \to 0$ when $|\mu_2| \to \infty$ and since $F_2(\mu) = \dfrac{h(\mu)}{\mu}$, the integral $(7)_e$ converges. Another integration would show the convergence to be absolute in this case ($f(x)$ continuous). This is also shown in the following.

In the general case we must consider singular points such as x_0 where $f(x)$ becomes discontinuous or infinite but in such a manner that $\int_{-\infty}^{\infty} |f(x)|\,dx$ converges. It is then sufficient to show that the integral $\int_{x_0-\rho}^{x_0+\rho} f(x)\, e^{\mu x} p_2^{\mu}(x+iy)\,dx$ (where ρ_0 is arbitrary positive constant) contributes to $F_2(\mu)$ a function for which the integral $(7)_e$ converges absolutely. Deleting all such singular points, the remaining path of the integral may be joined together so as to

make a continuous function whose contribution to $F_2(\mu)$ will make the integral like $(7)_c$ absolutely convergent.

If we take ρ small it is sufficient to show that the integral $\displaystyle\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{p_2^{\mu}(x_0+iy)}{\mu-\lambda}d\mu\int_{x_0-\rho}^{x_0+\rho}f(x)e^{\mu x}dx$ converges absolutely.

Or (for the sake of simplicity in the integration) it is sufficient to show the absolute convergence of the integral

$$\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{p_2^{\mu}(x_0+iy)}{\mu-\lambda}d\mu\int_{-\infty}^{\infty}f(x)e^{\mu x}dx \quad \text{where} \quad \begin{cases} f(x)=C_1(x-x_0)^{-K}e^{-(x-x_0)\delta} & \text{for } x_0<x \\ \quad\;\; =C_2e^{(x-x_0)\delta} & \text{for } x<x_0 \\ 0\le K<1 & \end{cases}$$

This represents the allowable type of infinities for $f(x)$. This integral is found to be

$$ie^{\mu_1x_0}\int_{-\infty}^{\infty}\frac{p_2^{\mu}(x_0+iy)}{\mu_1-\lambda+i\mu_2}\left[\frac{C_2}{\mu_1+\delta+i\mu_2}+\frac{C_1\Gamma(1-K)}{(\delta-\mu_1-i\mu_2)^{1-K}}\right]d\mu_2$$

which is seen to be absolutely convergent.

In the case $C_1=C_2$ and $K=0$, $f(x)$ is continuous, but $f'(x)$ not. The integral becomes

$$2\delta C_1ie^{\mu_1x_0}\int_{-\infty}^{\infty}\frac{(\cos\mu_2x_0+i\sin\mu_2x_0)}{(\mu_1-\lambda+i\mu_2)\left[(\mu_1+i\mu_2)^2-\delta^2\right]}d\mu_2$$ so that not

only $(7)_c$ but also $(7)_e$ converges absolutely.

The conclusion is that $F_1(\mu)$ and $F_2(\mu)$ are of class F.

Two theorems may now be proven.

Theorem A.

If we assume that any developable function $f(x)$ of class f may be represented for almost every x by the integral

$$9)_a \qquad f(x) = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\mu e^{-\mu x} p_2^{\mu}(x+iy)}{\Omega(0,\mu,\mu)} F(\mu)\, d\mu$$

where $F(\mu)$ is of class F, and developable with the same δ constant as $f(x)$, then the unique solution of this integral equation is

$$9)_b \qquad F(\mu) = F_2(\mu) \equiv \int_{-\infty}^{\infty} f(x) e^{\mu x} p_2^{\mu}(x+iy)\, dx$$

In other words if the representation of $f(x)$ of type $(9)_a$ is possible it must be identical with that given in $(6)_a$

Theorem B. The hypothesis $(9)_a$ is correct. Any function $f(x)$ of class f is representable for almost every x.

The reciprocal theorems may be proven in a manner so similar to the proof for these, that it is only neccessary to state them.

Theorem A'.

If instead of $(9)_a$ we assume

$$(9)'_a \qquad f(x) = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\mu e^{\mu x} p_2^{\mu}(x+iy)}{\Omega(0\mu,\mu)} F_1(\mu)\, d\mu$$

the unique solution is

$$(9)'_b \qquad F(\mu) = F_1(\mu) = \int_{-\infty}^{\infty} f(x) e^{-\mu x} p_1^{\mu}(x+iy)\, dx$$

so that if $(9)'_a$ is a possible representation it must be identical with $(6)_b$.

Theorem B'. Any developable function $f(x)$ of class f may be represented thus for almost every x.

When these theorems are proven, it then follows that the transforms of $F_L(\mu)$ defined by by $(9)_a$ or $(9)'_a$ are of class f when F is of class F and the pair of equations $(9)_a$ and $(9)_b$ are equivalent, each being the solution of the other.

Similarly each of $(9)'_a$ and $(9)'_b$ is the solution of the other. In other words any developable function of μ which is of class F may be represented by either of the integrals

$$10)_a \quad F(\mu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{-\mu x}\, p_1^{\mu}(x+iy)\,dx \int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\lambda\, e^{\lambda x}\, p_2^{\lambda}(x+iy)}{\Omega(0,\lambda,\lambda)}\, F(\lambda)\,d\lambda$$

$$10)_b \quad F(\mu) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} e^{\mu x}\, p_2^{\mu}(x+iy)\,dx \int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\lambda\, e^{-\lambda x}\, p_1^{\lambda}(x+iy)}{-\Omega(0,\lambda,\lambda)}\, F(\lambda)\,d\lambda$$

The system of four integral identities, consisting of these two and the two $(6)_a$ and $(6)_b$ constitute a generalization of the form $(2)_a$ and $(2)_b$ of Fourier's integral identity

To prove theorem A, change the notation $(9)_b$ replacing μ by λ

$$11)_a \quad F_2(\lambda) = \int_0^\infty f(x) e^{\lambda x} p_2^\lambda(x+iy)\,dx + \int_{-\infty}^0 f(x) e^{\lambda x} p_2^\lambda(x+iy)\,dx$$

It is to be proven, when $f(x)$ in these integrals is replaced by the assumed integral $(9)_a$ this leads to $F_2(\lambda) = F(\lambda)$ where $F_2(\lambda)$ is of class F by this definition and $F(\lambda)$ is by hypothesis. By the latter it is evident that the integral $(9)_a$ has the same value for every choice of path $\mu = \mu_1 =$ constant in the interval $e_0 < \mu_1 < \delta$ if $e_0 > 0$, or the interval $0 < \mu_1 < \delta$ if $e_0 \leq 0$. Hence let $a_2 \equiv \delta - \epsilon$ and $a_1 \equiv e_0 + \epsilon$ if $e_0 > 0$ or $a_1 = \epsilon$ if $e_0 \leq 0$ where ϵ is an arbitrary small positive constant. It will then be sufficient to prove that $F(\lambda) = F_2(\lambda)$ when $a_1 < \lambda_1 < a_2$.

Assuming $a_1 < \lambda_1 < a_2$, replace $f(x)$ in $(11)_a$ by the assumed integral representation $(9)_a$ giving μ_1 the permissible value a_2 in the first and the value a_1 in the second integral of $(11)_a$. Let

$$11)_b \quad I_+(x_0) \equiv \frac{1}{2\pi i}\int_0^{x_0} e^{\lambda x} p_2^\lambda(x+iy)\,dx \int_{a_2-i\infty}^{a_2+i\infty} 2\mu e^{-\mu x} p_1^\mu(x+iy)\frac{F(\mu)\,d\mu}{\Omega(0,\mu,\mu)}$$

$$11)_c \quad I_-(x_0) \equiv \frac{1}{2\pi i}\int_{-x_0}^{0} e^{\lambda x} p_2^\lambda(x+iy)\,dx \int_{a_1-i\infty}^{a_1+i\infty} 2\mu e^{-\mu x} p_1^\mu(x+iy)\frac{F(\mu)\,d\mu}{\Omega(0,\mu,\mu)}$$

Then eq (11)$_a$ may be written

$$11)_d \quad F_2(\lambda) = \lim_{x_0\to\infty}\left[I_+(x_0) + I_-(x_0)\right]$$

If interchanging the order of integration in these integrals gives absolutely convergent integrals the process is justified.

Since y is a constant $p_1^{''\mu}(x+iy) = D_z^2\, p_1^{\mu}(x+iy) = D_x^2\, p_1^{\mu}(x+iy)$ so the function $e^{-\mu x} p_1^{\mu}(x+iy)$ satisfies the differential equation (3) in which D_z^2 is replaced by D_x^2. Similarly $e^{\lambda x} p_2^{\lambda}(x+iy)$ satisfies (3) where D_z^2 is replaced by D_x^2 and μ^2 by λ^2. From these two equations we obtain the indefinite integral

$$11)_e \quad \int e^{-\mu x} p_1^{\mu}(x+iy)\, e^{\lambda x}\, p_2^{\lambda}(x+iy)\, dx =$$

$$= \frac{e^{(\lambda-\mu)x}}{\lambda^2-\mu^2}\left\{(\lambda+\mu) p_1^{\mu}(x+iy) p_2^{\lambda}(x+iy) + p_1^{\mu}(x+iy) p_2^{'\lambda}(x+iy) - p_1^{'\mu}(x+iy) p_2^{\lambda}(x+iy)\right\}$$

hence

$$11)_f \quad \int_0^{x_0} \left[e^{-\mu x} p_1^{\mu}(x+iy) e^{\lambda x} p_2^{\lambda}(x+iy)\right]_{\mu_1 = a_2} dx =$$

$$= \frac{1}{\mu^2-\lambda^2}\Big\{(\lambda+\mu) p_1^{\mu}(iy) p_2^{\lambda}(iy) + p_1^{\mu}(iy) p_2^{'\lambda}(iy) - p_1^{'\mu}(iy) p_2^{\lambda}(iy)$$

$$+ e^{-x_0[a_2-\lambda_1 + i(\mu_2-\lambda_2)]}\left[(\lambda+\mu) p_1^{\mu} p_2^{\lambda} + p_1^{\mu} p_2^{'\lambda} - p_1^{'\mu} p_2^{\lambda}\right]_{(x_0+iy)}\Big\}$$

and

$$(11)_g \quad \int_{-x_0}^{0}\left[e^{-\mu x} p_1^{\mu}(x+iy)\, e^{\lambda x} p_2^{\lambda}(x+iy)\right]_{\mu_1=a_1} dx =$$

$$= \frac{-1}{\mu^2-\lambda^2}\Big\{(\lambda+\mu)\, p_1^{\mu}(iy)\, p_2^{\lambda}(iy) + p_1^{\mu}(iy)\, p_2'^{\lambda}(iy) - p_1'^{\mu}(iy)\, p_2^{\lambda}(iy)$$

$$- e^{-x_0[\lambda_1-a_1+i(\lambda_2-\mu_2)]}\left[(\lambda+\mu)\, p_1^{\mu} p_2^{\lambda} + p_1^{\mu} p_2'^{\lambda} - p_1'^{\mu} p_2^{\lambda}\right]_{(-x_0+iy)}\Big\}$$

Reference to the assumptions C, D, C' and D' shows that since $0<a_1<\lambda_1<a_2$ the terms of these equations which contain x_0 will vanish when $x \to \infty$. Hence the result of interchanging the order of integration in $(11)_f$ and $(11)_e$ give

$$(11)_h \quad F_2(\lambda) = I_+(\infty) + I_-(\infty) =$$

$$= \frac{1}{2\pi i}\int_{a_2-i\infty}^{a_2+i\infty} \frac{2\mu\, F(\mu)}{(\mu^2-\lambda^2)\,\Omega(0,\mu,\mu)}\left[(\lambda+\mu)\, p_1^{\mu}(iy)\, p_2^{\lambda}(iy) + p_1^{\mu}(iy)\, p_2'^{\lambda}(iy) - p_1'^{\mu}(iy)\, p_2^{\lambda}(iy)\right] d\mu$$

$$- \frac{1}{2\pi i}\int_{a_1-i\infty}^{a_1+i\infty} \frac{2\mu\, F(\mu)}{(\mu^2-\lambda^2)\,\Omega(0,\mu,\mu)}\left[(\lambda+\mu)\, p_1^{\mu}(iy)\, p_2^{\lambda}(iy) + p_1^{\mu}(iy)\, p_2'^{\lambda}(iy) - p_1'^{\mu}(iy)\, p_2^{\lambda}(iy)\right] d\mu$$

Reference to B, B' and $(5)_d$ shows that when $\mu_2 \to \pm\infty$, the integrand of these integrals vanishes, for it becomes $\frac{F(\mu)}{\mu-\lambda}$ multiplied by the constant $p_2^{\lambda}(iy)$. Since

$F(\mu)$ is by hypothesis of class F these integrals converge absolutely, thus justifying the order change. The two integrals in $(11)_h$ are equivalent to a contour integral with one simple pole at the point $\mu = \lambda$ in its interior.

Hence, when $a_1 < \lambda_1 < a_2$, this becomes

$$11)_i \quad F_2(\lambda) = \frac{F(\lambda)}{\Omega(0,\lambda,\lambda)}\left[2\lambda\, p_1^{\lambda}(iy)\, p_2^{\lambda}(iy) + p_1^{\lambda}(iy)\, p_2'^{\lambda}(iy) - p_1'^{\lambda}(iy)\, p_2^{\lambda}(iy)\right]$$

$$= \frac{\Omega(iy,\lambda,\lambda)}{\Omega(0,\lambda,\lambda)} F(\lambda) = F(\lambda) \qquad \text{by } (5)_e$$

This establishes theorem A. The proof of theorem A' is precisely the same.

To prove theorem B it must be shown that the double integral in $(6)_a$, for all values of x for which $f(x)$ is not infinite, converges to a function, say $I(x)$, which is equal to $f(x)$ for almost every x.

Assuming that x is a value for which $f(x)$ is not infinite, let

$$12)_a \quad I(x) = \lim_{\eta\to\infty}\ \lim_{x_0\to\infty} I(x, x_0, \eta)$$

where

$$12)_b \quad I(x,x_0,\eta) \equiv \frac{1}{2\pi i}\int_{\mu_1-i\eta}^{\mu_1+i\eta} \frac{2\mu e^{-\mu x} p_1^{\mu}(x+iy)}{\Omega(0,\mu,\mu)}\,d\mu \int_{-x_0}^{x_0} f(x_1)\, e^{\mu x_1} p_2^{\mu}(x_1+iy)\,dx_1$$

where $c_0 < \mu_1 < \delta$

The limit $x_0 \to \infty$ of this x_1 integral is the transform $F_2(\mu)$ of $f(x)$ and it has been shown to converge absolutely and also uniformly as to μ_2 in the interval $-\eta < \mu_2 < \eta$.

Therefore

$$12)_c \quad I(x,x_0,\eta) = \frac{1}{2\pi i}\int_{-x_0}^{x_0} f(x_1)\,dx_1 \int_{\mu_1-i\eta}^{\mu_1+i\eta} \frac{2\mu e^{\mu(x_1-x)} p_1^{\mu}(x+iy)\, p_2^{\mu}(x_1+iy)}{\Omega(0,\mu,\mu)}\,d\mu$$

Consequently a necessary and sufficient condition that the integral $(6)_a$ converges to a limit $I(x)$ is that $\lim_{x_0\to\infty} \lim_{\eta\to\infty} I(x,x_0,\eta)$ exists $= I(x)$.

By the condition of developability of $f(x)$ eq $(5)_g$, $c_0 < \delta$, so the integrand of the μ integral in $(12)_c$, say $\Psi(x,x_1,\mu)$, is for any fixed values of x, x_1 ady an analytic function of μ in the half plane $c_0 < \mu_1 < \delta$. Hence this integral is equal to

$\int_{\mu_1-i\eta}^{\mu_1+i\eta} \Psi(x,x_1,\mu)\,d\mu$ taken between the same limits but

along a semi-circular path of radius η (on the right),

Now

$$\Psi(x, x_1, \mu) \equiv \frac{e^{\mu(x_1-x)} \cdot 2\mu\, P_1^{\mu}(x+iy)\, P_2^{\mu}(x+iy)}{\Omega(0\mu, \mu)} \rightarrow e^{\mu(x_1-x)} \text{ when } |\mu| \rightarrow \infty$$

Hence

12)$_d$ $$\lim_{x_0\to\infty} \lim_{\eta\to\infty} I(x, x_0, \eta) = \lim_{x_0\to\infty} \lim_{\eta\to\infty} \frac{1}{2\pi i} \int_{-x_0}^{x_0} f(x_1)\,dx_1 \int_{\mu_1-i\eta}^{\mu_1+i\eta} e^{\mu(x_1-x)}\, d\mu$$

$$= \lim_{x_0\to\infty} \lim_{\eta\to\infty} \frac{1}{2\pi i} \int_{-x_0}^{x_0} f(x_1) \left[\frac{e^{(\mu_1+i\eta)(x_1-x)} - e^{(\mu_1-i\eta)(x_1-x)}}{x_1 - x} \right] dx_1$$

$$= \lim_{x_0\to\infty} \lim_{\eta\to\infty} \frac{1}{\pi} \int_{-x_0}^{x_0} e^{\mu_1(x_1-x)} f(x_1) \sin \frac{\eta(x-x_1)}{x-x_1}\, dx_1$$

$$= \lim_{t_0\to\infty} \lim_{\eta\to\infty} \frac{1}{\pi} \int_{+0}^{t_0} \left[e^{\frac{\mu_1 t}{\eta}} f(x+\tfrac{t}{\eta}) + e^{-\frac{\mu_1 t}{\eta}} f(x-\tfrac{t}{\eta}) \right] \frac{\sin \eta t}{t}\, dt$$

Since the constant μ_1 is less than δ, this bracket vanishes exponentially when $t \to \infty$, and it is absolutely integrable. The condition (1)$_a$ also implies that it is of limited variation in any finite interval. Hence by Dirichlet's theorem we obtain

12)$_e$ $I(x) = \frac{1}{2}\left[f(x+0) + f(x-0) \right]$ since, by hypothesis, x

is a value for which $f(x)$ is not infinite. This

proves theorem B and the proof of theorem B' is similar.

If $g(x)$ is of class f with the same δ and $G_1(\mu), G_2(\mu)$ its transforms, the analogue of Parseval's formula is (formally

$$13)_a \quad \int_{-\infty}^{\infty} f(x)g(x)dx = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta+0+i\infty} \frac{2\mu F_1(\mu)G_2(\mu)}{\Omega(0,\mu,\mu)}d\mu = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0-i\infty} \frac{2\mu F_2(\mu)G_1(\mu)}{\Omega(0,\mu,\mu)}d\mu$$

$$13)_b \quad \int_{-\infty}^{\infty} f^2(x)dx = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\mu F_1(\mu)F_2(\mu)}{\Omega(0,\mu,\mu)}d\mu$$ provided that

f and g are further restricted so that these integrals converge.

If we let $\xi = e^{x}$ and $f(x) = F(\xi)$; the eq (6)$_a$ with $y = 0$ becomes

$$14) \quad F(\xi) = \frac{1}{2\pi i}\int_{\delta-0-i\infty}^{\delta-0+i\infty} \frac{2\mu\,\xi^{-\mu}\,p_1^{\mu}(\log\xi)}{\Omega(0,\mu,\mu)}d\mu \int_0^{\infty} F(\xi_1)\,\xi_1^{\mu-1}\,p_2^{\mu}(\log\xi_1)\,d\xi_1$$

for $0 < \xi < \infty$.

In the degenerate case $p_1^{\mu} \equiv p_2^{\mu} \equiv 1$, $\Omega(0,\mu,\mu) \equiv 2\mu$ and this becomes Mellin's form of Fourier's integral.

Some implied limitations may be removed so that in case $c_0 < 0$ the integrals (6) may be taken up the imaginary axis of μ, thus bringing them into closer analogy with the trigonometric form of Fourier's integral. In this case the exponential vanishing of $f(x)$ when $x \to \pm\infty$ is no longer necessary being replaceable by mere vanishing. $(\delta = 0)$.

The conditions A, B, C, D, and A', B', C', D', imposed upon the solutions of equ(3) were limited to the half-plane, $0 < \mu_1$, for the sake of brevity.

In the applications to Cylinder functions it is found that these conditions apply throughout the μ-plane with a cut from the origin to ∞.

Another case is where the given function $q(z)$ in the differential equation (3) vanishes like

15) $q(z) \sim c e^{-2c|x|}$ as $x \to \pm\infty$ where c is a positive constant.

The conditions imposed on C and C' are compatible for any value of μ, but taken together with A, B, A', and B' and the differential equation, they imply that $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ as functions of μ may have the point $\mu = -c$ on the negative real axis, as a singularity.

In that case, instead of limiting the fundamental assumptions to the half plane $(4)_b$, $0<\mu_1<\infty$, we could have imposed them upon the half plane

16)$_a$ $\quad -c<\mu_1<\infty \qquad -\infty<\mu_2<\infty \qquad$ where $c>0$

All of the assumptions remain valid for this half-plane except D and D' which should take the more comprehensive form, for $-c<\mu_1<c$,

D) $\quad p_1^\mu(z) \sim \frac{1}{2\mu}\left[\Omega(0,\mu,\mu)-\Omega(0,\mu,-\mu)e^{-2\mu z}\right]\left[1+\text{Zero } e^{-2cz}\right]$ if $x\to-\infty$

D') $\quad p_2^\mu(z) \sim \frac{1}{2\mu}\left[\Omega(0,\mu,\mu)-\Omega(0,-\mu,\mu)e^{-2\mu z}\right]\left[1+\text{Zero } e^{-2cz}\right]$ if $x\to+\infty$

which of course reduce to the first form when restricting μ to $0<\mu_1$. In the strip of the μ-plane defined by

16)$_b$ $\quad -c<\mu_1<c \qquad -\infty<\mu_2<\infty$

all four solutions of (3) $E_1^\mu(z)$, $E_2^\mu(z)$, $E_1^{-\mu}(z)$ and $E_2^{-\mu}(z)$ have a meaning so that $\Omega(0,\mu,-\mu)$ and $\Omega(0,-\mu,\mu)$ are analytic in this strip. The extension of the scope of the assumptions defining $E_1^\mu(z)$ and $E_2^\mu(z)$ to the half plane $(16)_a$ amounts to defining $E_1^{-\mu}(z)$ and $E_2^{-\mu}(z)$ for the half plane $-\infty>\mu_1>c$ which has the strip $(16)_b$ in common with the half plane $16)_a$. In this

common domain we find

17)$_a$ $\Omega(0,\mu,\mu)E_1^{-\mu}(z) = \Omega(0,-\mu,\mu)E_1^{\mu}(z) + 2\mu E_2^{\mu}(z)$

17)$_b$ $\Omega(0,\mu,\mu)E_2^{-\mu}(z) = \qquad 2\mu E_1^{\mu}(z) + \Omega(0,\mu,-\mu)E_2^{\mu}(z)$

whose solution for $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ is found by changing the sign of μ in these. There is the identity

18) $\Omega(0,\mu,\mu)\Omega(0,-\mu,-\mu) = \Omega(0,\mu,-\mu)\Omega(0,-\mu,\mu) - 4\mu^2$

and corresponding to (5)$_d$ which remains valid we find the additional relations

$$\left.\begin{array}{ll} 19)_a & e^{2\mu_2 y}\Omega(0,\mu,-\mu) \to 0 \\ 19)_b & e^{2\mu_2 y}\Omega(0,-\mu,\mu) \to 0 \end{array}\right\} \text{ when } \mu_2 \to \pm\infty \text{ and } -b < y < b$$

In case $q(z)$ is an even function of z, $E_2^{\mu}(z)$ becomes $E_1^{\mu}(-z)$ and in that case $\Omega(0,\mu,-\mu) = \Omega(0,-\mu,\mu)$

This extension does not affect the argument by which the integral representations (6)$_a$ and (6)$_b$ were obtained. It merely extends the domain of definition of $E_1^{\mu}(z)$ and $E_2^{\mu}(z)$ to the left to $\mu_1 = -c+0$, so that the strip of the μ-plane in which the transforms, $F_1(\mu)$ and $F_2(\mu)$ of $f(x)$, are analytic is bounded on the left by the greater of $\mu_1 = -c$ or $\mu_1 = -\delta$.

2 Application to Associated Legendre Functions.

In the applications in D. Debcos, the problem of finding a potential inside or outside a one-sheeted hyperboloid of revolution (of oblate spheroidal coordinates) when the potential has assigned values on the hyperboloid requires the integral representation of the given function $f(\beta)$ for $-\infty<\beta<\infty$ in terms of the function $v(\beta)=T^{\mu}_{m-\frac{1}{2}}(\tanh\beta)$ which satisfies the differential equation

20)$_a$ $$D^2_\beta v+\left[(m^2-\tfrac{1}{4})\operatorname{sech}^2\beta-\mu^2\right]v=0$$

The same problem for potential given on one sheet of the two-sheeted hyperboloids of prolate spheroidal coordinates, or any of its inversions, requires an integral representation for the positive range only, $0<\beta<\infty$, in terms of $v(\beta)=P^{\mu}_{m-\frac{1}{2}}(\coth\beta)$ or $Q^{\kappa}_{m-\frac{1}{2}}(\coth\beta)$ which satisfies

20)$_b$ $$D^2_\beta v+\left[(\tfrac{1}{4}-m^2)\operatorname{csch}^2\beta-\mu^2\right]v=0$$

The same equation with the same range is required for the corresponding problem with Toroidal coordinates.

The two preceding equations are included in the following special case of (3) where $z = x + iy$ and $E' = \mathcal{D}_z E$

21) $E''(z) + [(\nu^2 - \frac{1}{4})\operatorname{sech}^2 z - \mu^2] E(z) = 0$

where, since the constant parameter ν occurs only as ν^2 we may without loss of generality understand by ν

21)$_a$ $\nu = \nu_1 + i\nu_2$ where $0 \leq \nu_1$

The function $(\nu^2 - \frac{1}{4})\operatorname{sech}^2 z$ is an even function of z which is analytic in a strip of the z-plane

21)$_b$ $-b < y < b \equiv \frac{\pi}{2}$ and $-\infty < x < \infty$

It vanishes like $e^{-2|x|}$ when $x \to \pm\infty$ ($c = 1$)

If we let

22)$_a$ $\zeta = \tanh z$, then $E'(z) = (1-\zeta^2)\mathcal{D}_\zeta E$, $\frac{1-\zeta}{2} = \frac{1}{1+e^{2z}}$, $\frac{1+\zeta}{2} = \frac{1}{1+e^{-2z}}$

Eq(21) becomes

22)$_b$ $\mathcal{D}_\zeta [(1-\zeta^2)\mathcal{D}_\zeta E] + [(\nu^2 - \frac{1}{4}) - \frac{\mu^2}{1-\zeta^2}] E = 0$ which has

solutions $E = T^{\mu}_{\nu-\frac{1}{2}}(\zeta)$, $T^{\mu}_{\nu-\frac{1}{2}}(-\zeta)$, $T^{-\mu}_{\nu-\frac{1}{2}}(\zeta)$ etc.

If we take $z = x = \beta =$ real and $\nu = m$ eq (21) becomes (20)$_a$

Taking $y = \pm(\frac{\pi}{2} - 0)$ so that $z = x \pm i\frac{\pi}{2}$ equ(21) reduces to (20)$_b$ which is not of the form (3) since (20)$_b$ has a singularity at $\beta = 0$. However, in the integral

representations to be found in terms of solutions of (21) we may obtain integrals suitable for (20)$_b$ by placing suitable restrictions upon the nature of the function $f(x)$ near $x=0$, and thus make use of the limiting case $y = \pm \pi/2$.

As two fundamental solutions of (21) we may take

23)$_a$ $$E_1^{\mu}(z) = E_\nu^{\mu}(z) \equiv e^{-\mu z} p_\nu^{\mu}(z) \equiv \frac{\Gamma(1+\mu)\Gamma(\frac{1}{2}+\nu-\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)} T_{\nu-\frac{1}{2}}^{\mu}(\tanh z)$$

23)$_b$ $$E_2^{\mu}(z) = E_\nu^{\mu}(-z) \equiv e^{\mu z} p_\nu^{\mu}(-z)$$

From the definition of $T_{\nu-\frac{1}{2}}^{\mu}(\zeta)$ as a hypergeometric function with argument $\frac{1-\zeta}{2} = \frac{1}{1+e^{2z}}$ we find

24)$_a$ $$p_\nu^{\mu}(z) \equiv F\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu, 1+\mu; \frac{1}{1+e^{2z}}\right)$$ or by Gauss's transformation

24)$_b$ $$p_\nu^{\mu}(z) \equiv \frac{-\cos\nu\pi}{\sin\mu\pi} e^{2\mu z} F\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu, 1+\mu; \frac{1}{1+e^{-2z}}\right)$$
$$+ \frac{\Gamma(\mu)\Gamma(1+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)\Gamma(\frac{1}{2}-\nu+\mu)} F\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu, 1-\mu; \frac{1}{1+e^{-2z}}\right)$$

24)$_c$ $$p_\nu^{0}(z) = -\frac{2\cos\nu\pi}{\pi}\left[z - \tfrac{1}{2}\log(1+e^{2z})\right] F\left(\tfrac{1}{2}+\nu, \tfrac{1}{2}-\nu, 1; \frac{1}{1+e^{-2z}}\right)$$
$$- \frac{\cos^2\nu\pi}{\pi^2}\sum_{s=0}^{\infty} \frac{1}{(1+e^{-2z})^s} \frac{\Gamma(s+\frac{1}{2}+\nu)\Gamma(s+\frac{1}{2}-\nu)}{\Gamma^2(s+1)}\left[\psi(s+\tfrac{1}{2}+\nu) + \psi(s+\tfrac{1}{2}-\nu) - 2\psi(s+1)\right]$$

If z is in the strip $(21)_b$ the hypergeometric function in $(24)_a$ converges for $-\infty < x \leq \infty$ provided that μ is in the half-plane,

$-1 < \mu_1 < \infty$ and $-\infty < \mu_2 < \infty$.

The nature of P_ν^μ when $x \to -\infty$ is shown by $(24)_a$ if $0 < \mu_1$ and more generally by $(24)_b$ and $(24)_c$ for $-1 < \mu_1$

The relation VI $(28)_a$ becomes with $\zeta = \tanh z$

$$25)\quad E_\nu^\mu(z) = \frac{\Gamma(1+\mu)\,\Gamma(\frac{1}{2}+\nu-\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\, T^\mu_{\nu-1/2}(\zeta) =$$

$$= \frac{2^{\mu-1}\Gamma(1+\mu)(1-\zeta^2)^{\frac{\mu}{2}}}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}\left\{ f\left(\tfrac{1}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2},\ \tfrac{1}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2},\ \tfrac{1}{2};\ \zeta^2\right) - \zeta f\left(\tfrac{3}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2},\ \tfrac{3}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2},\ \tfrac{3}{2};\ \zeta^2\right)\right\}$$

whence

$$E_\nu^\mu(0) = \frac{\sqrt{\pi}\,\Gamma(1+\mu)}{2^\mu\,\Gamma(\frac{3}{4}+\frac{\nu}{2}+\frac{\mu}{2})\,\Gamma(\frac{3}{4}-\frac{\nu}{2}+\frac{\mu}{2})} \quad\text{and}\quad E_\nu^{\prime\mu}(0) = \frac{-2\sqrt{\pi}\,\Gamma(1+\mu)}{2^\mu\,\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\mu}{2})\,\Gamma(\frac{1}{4}-\frac{\nu}{2}+\frac{\mu}{2})}$$

and

$$26)_a\quad \Omega(z,\mu,\mu) = \Omega(0,\mu,\mu) \equiv E_\nu^\mu(z)\, D_z\, E_\nu^\mu(-z) - E_\nu^{\prime\mu}(z)\, E_\nu^\mu(-z) = -2E_\nu^\mu(0)\, E_\nu^{\prime\mu}(0) =$$

$$= \frac{2\,\Gamma^2(1+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}$$

Hence

26)$_b$ $\dfrac{\Omega(0,\mu,\mu)}{2\mu} = \dfrac{\Gamma(\mu)\,\Gamma(\mu+1)}{\Gamma(\frac{1}{2}+\nu+\mu)\,\Gamma(\frac{1}{2}-\nu+\mu)}$ so $\Omega(0,0,0) = \dfrac{2\cos\nu\pi}{\pi} = \dfrac{2}{\Gamma(\frac{1}{2}+\nu)\,\Gamma(\frac{1}{2}-\nu)}$

26)$_c$ $\Omega(0,-\mu,\mu) = \Omega(0,\mu,-\mu) = \dfrac{2\mu\cos\nu\pi}{\sin\mu\pi}$

From (24) it is evident that, if z is in the strip (21)$_e$ and μ in the half plane $-1 < \mu$,

A) $p_\nu^\mu(z)$ is an analytic function of z and of μ
B) $p_\nu^\mu(z) \sim 1 + \text{Zero}\,\frac{1}{\mu}$ when $|\mu| \to \infty$ } for $-\infty < x \leq \infty$
C) $p_\nu^\mu(z) \sim 1 + \text{Zero}\,e^{-2z}$ when $x \to +\infty$

D) $p_\nu^\mu(z) \sim \frac{1}{2\mu}\left[\Omega(0,\mu,\mu) - \Omega(0,-\mu,\mu)\,e^{2\mu z}\right]\left[1 + \text{Zero}\,e^{2z}\right]$ when $x \to -\infty$

or

D)$'$ $p_\nu^0(z) \sim -z\,\Omega(000) + \text{finite terms}$ when $x \to -\infty$

Since $p_2^\mu(z) = p_\nu^\mu(-z)$ all the conditions A$'$ B$'$ C$'$ and D$'$ are satisfied so that $E_1^\mu = E_\nu^\mu(z)$ and $E_2^\mu = E_\nu^\mu(-z)$ satisfy the conditions necessary and sufficient for the integral representations (6)$_a$ and (6)$_e$.

Since $\nu_1 \geq 0$ it is evident that there is no zero of (26)$_f$ whose real part is greater than $\nu_1 - \frac{1}{2}$ and by (21)$_a$

$-\frac{1}{2} < c_0 \equiv \nu_1 - \frac{1}{2}$

The equation

$$27)_a \quad T^{\mu}_{\nu-1/2}(-\zeta) = \frac{1}{\sin\mu\pi}\left\{-\cos\nu\pi\, T^{\mu}_{\nu-1/2}(\zeta) + \cos(\mu-\nu)\pi\,\frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(\frac{1}{2}+\nu-\mu)}\, T^{-\mu}_{\nu-1/2}(\zeta)\right\}$$

may be written

$$27)_b \quad \frac{E^{\mu}_{\nu}(-z)}{\Gamma(1+\mu)} = \frac{1}{\sin\mu\pi}\left\{\frac{-\cos\nu\pi}{\Gamma(1+\mu)}\,E^{\mu}_{\nu}(z) + \frac{\pi}{\Gamma(\frac{1}{2}+\nu+\mu)\Gamma(\frac{1}{2}-\nu+\mu)}\,\frac{E^{-\mu}_{\nu}(z)}{\Gamma(1-\mu)}\right\}$$

The points of the μ-plane where T^{μ}_{ν} and $T^{-\mu}_{\nu}$ become linearly dependant are at $\mu = s =$ any real integer. The function $E^{\mu}_{\nu}(z)$ becomes infinite if μ is a negative integer, but $E^{\mu}_{\nu}(z)/\Gamma(1+\mu)$ is an integral function of μ. and $(27)_a$ gives

$$27)_c \quad T^{-s}_{\nu-\frac{1}{2}}(\zeta) = (-1)^s\,\frac{\Gamma(\frac{1}{2}+\nu-s)}{\Gamma(\frac{1}{2}+\nu+s)}\,T^{s}_{\nu-\frac{1}{2}}(\zeta) \quad \text{where } s \text{ is any integer.}$$

In the degenerate case $\nu = \pm\frac{1}{2}$, $\Omega(0,\mu,\mu)/2\mu \equiv 1$ but in all other cases there are zeros of this function where $T^{\mu}_{\nu-\frac{1}{2}}(\zeta)$ and $T^{\mu}_{\nu-\frac{1}{2}}(-\zeta)$ become linearly dependant.

$$28)\begin{cases} \dfrac{\Omega(0,\mu,\mu)}{2\mu} \equiv \dfrac{\Gamma(\mu)\,\Gamma(\mu+1)}{\Gamma(\frac{1}{2}+\nu+\mu)\Gamma(\frac{1}{2}-\nu+\mu)} = 0 \quad \text{when} \\ \left.\begin{array}{l} \mu = \mu_t = \pm\nu - \frac{1}{2} - t \\ \text{and} \\ T^{\mu_t}_{\nu-1/2}(-\zeta) = (-1)^t\, T^{\mu_t}_{\nu-1/2}(\zeta) \end{array}\right\} \quad t = 0, 1, 2, 3, \ldots\ldots \infty \end{cases}$$

Let $C_\nu^\mu(\zeta)$ be the even, and $S_\nu^\mu(\zeta)$ the odd function of ζ defined by

29)$_a$ $C_\nu^\mu(\zeta) \equiv$

$$\equiv 2^{\mu-1}\pi^{-3/2}\cos(\mu-\nu)\pi\,\Gamma\!\left(\tfrac{1}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2}\right)\Gamma\!\left(\tfrac{1}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2}\right)(1-\zeta^2)^{\mu/2}\,F\!\left(\tfrac{1}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2},\tfrac{1}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2},\tfrac{1}{2};\zeta^2\right)$$

29)$_b$ $S_\nu^\mu(\zeta) \equiv$

$$\equiv 2^{\mu}\pi^{-3/2}\cos(\mu-\nu)\pi\,\Gamma\!\left(\tfrac{3}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2}\right)\Gamma\!\left(\tfrac{3}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2}\right)\cdot\zeta(1-\zeta^2)^{\mu/2}\,F\!\left(\tfrac{3}{4}+\tfrac{\nu}{2}+\tfrac{\mu}{2},\tfrac{3}{4}-\tfrac{\nu}{2}+\tfrac{\mu}{2};\tfrac{3}{2};\zeta^2\right)$$

Equ(25) is the same as

29)$_c$ $T_{\nu-1/2}^\mu(\zeta) = C_\nu^\mu(\zeta) - S_\nu^\mu(\zeta)$ and $T_{\nu-1/2}^\mu(-\zeta) = C_\nu^\mu(\zeta) + S_\nu^\mu(\zeta)$

29)$_d$ If $\nu \to \pm\frac{1}{2}$, $T_{\nu-1/2}^\mu(\tanh z) \to \dfrac{\pm e^{-\mu z}}{\Gamma(1-\mu)}$ so $C_\nu^\mu(\tanh z) \to \dfrac{\pm\cosh\mu z}{\Gamma(1-\mu)}$, $S_\nu^\mu(\tanh z) \to \dfrac{\pm\sinh\mu z}{\Gamma(1-\mu)}$

The function $f(x)$ to be developed satisfies;

30)$_a$ $\int_{-\infty}^{\infty}|f(x)|\,dx$ converges, and $\lim\limits_{x\to\pm\infty} e^{\kappa|x|}f(x) = 0$ if $\kappa < \delta$ where

30)$_b$ $\nu_1 - \frac{1}{2} < \delta$ also $0 < \delta$

The integral identities (6) hold, if the path is $\mu = \mu_1$ where

30)$_c$ $-\delta < \mu_1 < \delta$ and $\nu_1 - \frac{1}{2} < \mu_1$

If $\nu_1 > \frac{1}{2}$ these reduce to the condition $\nu_1 - \frac{1}{2} < \mu_1 < \delta$

The constant y_0 must be in the interval

30)$_d$ $-\frac{\pi}{2} < y_0 < \frac{\pi}{2}$

The integral identities (6) may then be put in the

following forms (for $-\infty < x < \infty$)

$$31)_a \quad f(x)=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\mu\pi}{\cos(\mu-\nu)\pi}\frac{\Gamma(\frac{1}{2}+\nu-\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\mathrm{T}^{\mu}_{\nu-1/2}(\tanh(x+iy))\,d\mu\int_{-\infty}^{\infty}f(x_1)\,\mathrm{T}^{\mu}_{\nu-1/2}(-\tanh(x_1+iy))\,dx_1$$

$$31)_b \quad f(x)=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\mu\pi}{\cos(\mu-\nu)\pi}\frac{\Gamma(\frac{1}{2}+\nu-\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\mathrm{T}^{\mu}_{\nu-1/2}(-\tanh(x+iy))\,d\mu\int_{-\infty}^{\infty}f(x_1)\,\mathrm{T}^{\mu}_{\nu-1/2}(\tanh(x_1+iy))\,dx_1$$

$$31)_c \quad f(x)=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\mu\pi}{\cos(\mu-\nu)\pi}\frac{\Gamma(\frac{1}{2}+\nu-\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\,d\mu\int_{-\infty}^{\infty}f(x_1)\Big[\mathrm{C}^{\mu}_{\nu}(\tanh(x+iy))\,\mathrm{C}^{\mu}_{\nu}(\tanh(x_1+iy))$$
$$-\mathrm{S}^{\mu}_{\nu}(\tanh(x+iy))\,\mathrm{S}^{\mu}_{\nu}(\tanh(x_1+iy))\Big]\,dx_1$$

With $y=0$, and $\nu=\pm\frac{1}{2}$ the latter becomes by $(29)_d$

$$f(x)=\frac{1}{2\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}d\mu\int_{-\infty}^{\infty}f(x_1)\cosh\mu(x-x_1)\,dx_1=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\mu_2\int_{-\infty}^{\infty}f(x_1)\cos\mu_2(x-x_1)\,dx_1$$

$$=\frac{1}{\pi}\int_{0}^{\infty}d\nu\int_{-\infty}^{\infty}f(x_1)\cos\nu(x-x_1)\,dx_1 \quad \text{which is Fourier's integral.}$$

Eqn $(31)_c$ may be obtained directly from $(31)_a$ by using $(27)_a$ and then moving the path of the integral containing $\mathrm{T}^{-\mu}$ from μ_1 to $-\mu_1$ and then reversing the sign of the variable of integration to recover the original path. The sum of the residuals of the poles of $\mu\pi/\sin\mu\pi$ thus passed over is zero by $(27)_c$. By making use of both or all three

forms of (31) one avoids repeating this transformation of integrals in various applications

Eq. (31)a could be put in the following form (taking $y=0$, and $\xi = \tanh x$)

$$31)_d \quad F(\xi) = \frac{1}{2\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu\pi\,\Gamma(\mu-\nu)\,T_\nu^\mu(\xi)}{\sin(\mu-\nu)\pi\,\Gamma(\mu-\nu+1)}\,d\mu \int_{-1}^{1} \frac{F(\xi_1)}{1-\xi_1^2}\,T_\nu^\mu(-\xi_1)\,d\xi_1 \quad \text{for } -1<\xi<1$$

where $-\delta < \mu_1 < \delta$ and $\nu_1 < \mu_1$ and $\nu_1 < \delta$ and

$\int_{-1}^{1} \frac{|F(\xi)|\,d\xi}{1-\xi^2}$ converges and $\lim\limits_{\xi\to\pm 1} (1-\xi^2)^{\mu_1} F(\xi) = 0$ if $\mu_1 < \frac{\delta}{2}$

Eq. (31)e may also be put in the form

$$31)_e \quad f(x) = \frac{-1}{4\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu\sin\mu\pi\,d\mu \int_{-\infty}^{\infty} f(x_1)\left\{\frac{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(\frac{1}{4}+\frac{\nu}{2}-\frac{\mu}{2})\,\mathcal{C}_\nu^\mu(\tanh(x+iy))\,\mathcal{C}_\nu^\mu(\tanh(x_1+iy))}{\Gamma(\frac{3}{4}+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\mu}{2})\,(\cos\mu\pi - \sin\nu\pi)} + \frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{\mu}{2})\,\mathcal{S}_\nu^\mu(\tanh(x+iy))\,\mathcal{S}_\nu^\mu(\tanh(x_1+iy))}{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{\mu}{2})\Gamma(\frac{1}{4}+\frac{\nu}{2}-\frac{\mu}{2})\,(\cos\mu\pi + \sin\nu\pi)}\right\}dx_1$$

where

$$31)_f \quad \begin{cases} \mathcal{C}_\nu^\mu(\xi) \equiv (1-\xi^2)^{\frac{\mu}{2}}\,F\left(\frac{1}{4}+\frac{\nu}{2}+\frac{\mu}{2},\ \frac{1}{4}-\frac{\nu}{2}+\frac{\mu}{2},\ \frac{1}{2};\ \xi^2\right) \\ \mathcal{S}_\nu^\mu(\xi) \equiv 2\xi(1-\xi^2)^{\frac{\mu}{2}}\,F\left(\frac{3}{4}+\frac{\nu}{2}+\frac{\mu}{2},\ \frac{3}{4}-\frac{\nu}{2}+\frac{\mu}{2},\ \frac{3}{2};\ \xi^2\right) \end{cases}$$

These are even integral functions of μ and of ν.

Integral identities for the positive range $0 < x < \infty$.

If one is only concerned with positive values of x, then by placing $f(x) \equiv 0$ for $-\infty < x < 0$ any of the forms (31) apply or assuming an even or an odd [illegible] of x one gets two kinds of integrals from $(31)_c$ analogous to Fourier's cosine and sine integrals. It is worth noting that when we place $f(x) \equiv 0$ for $-\infty < x < 0$ the path of $(31)_\ell$ may [illegible] half plane determined by the two inequalities $-\delta < \mu_1 < \infty$ and $\nu_1 - \frac{1}{2} < \mu_1 < \infty$. It is evident that with this form the condition that $f(x)$ be developable (which was $\nu_1 - \frac{1}{2} < \delta$) is no longer neccessary and we could take $\delta = 0$. In other words instead of requiring the exponential vanishing of $f(x)$ when $x \to +\infty$, we now require only such vanishing as will secure the convergence of the x_1 integral in $(31)_\ell$. For example if $\nu_1 < \frac{1}{2}$ the path may be taken up the imaginary axis of μ, the integral $(31)_\ell$ then being valid if $f(x) \to 0$ when $x \to \infty$, as in Fourier's integral.

We next get a variety of forms for the positive range only, by considering the cases of $(31)_a$ and $(31)_b$ in which $y \to \pm(\frac{\pi}{2}-0)$. For this we place the additional restriction upon $f(x)$

32) $f(x) \sim C x^{\delta_0}$ when $x \to +0$ where $-1 < \delta_0$ in order that $\int_0^\infty |f(x)|dx$ will converge, and also $\nu_1 - \frac{3}{2} < \delta_0$ in order that integrals like $\int_0^b f(x) P^{\mu}_{\nu-\frac{1}{2}}(\coth x)dx$ may converge.

The necessity for this restriction is apparent from reference to formulas VI (43) and (45) which become for positive x

$$33)_a \quad P^{\mu}_{\nu-\frac{1}{2}}(\coth x) = \frac{\Gamma(\frac{1}{2}+\nu+\mu)(1-e^{-2x})^{\frac{1}{2}-\nu}}{\Gamma(\frac{1}{2}+\nu-\mu)\Gamma(\mu+1)} e^{-\mu x} F(\tfrac{1}{2}-\nu, \tfrac{1}{2}-\nu+\mu, \mu+1; e^{-2x})$$

$$33)_b \quad Q^{\mu}_{\nu-\frac{1}{2}}(\coth x) = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\nu+\mu)\cos\mu\pi\,(1-e^{-2x})^{\frac{1}{2}+\nu}}{2^{\nu+2}\,\Gamma(\nu+1)} e^{\mu x} F(\tfrac{1}{2}+\nu, \tfrac{1}{2}+\nu-\mu, 2\nu+1; 1-e^{-2x})$$

When $x \to 0$ Q vanishes (since we consider $\nu_1 \geq 0$) like

$33)_c$ $Q^{\mu}_{\nu-\frac{1}{2}}(\coth x) \sim \dfrac{x^{\frac{1}{2}+\nu}\sqrt{\pi}\,\Gamma(\frac{1}{2}+\nu+\mu)}{2\sqrt{2}\;\Gamma(\nu+1)}\cos\mu\pi$, but $P^{\mu}_{\nu-\frac{1}{2}}(\coth x)$ becomes infinite in general (if $\nu_1 > \frac{1}{2}$) like

$$33)_d \quad P^{\mu}_{\nu-\frac{1}{2}}(\coth x) \sim \frac{x^{\frac{1}{2}-\nu}\, 2^{\nu-\frac{1}{2}}\,\Gamma(\nu)}{\sqrt{\pi}\,\Gamma(\frac{1}{2}+\nu-\mu)} \quad \text{as } x \to +0$$

Write equations $(31)_a$ and $(31)_b$ first with y and then with $-y$, where $f(x) \equiv 0$ for $-\infty < x < 0$, and in each of the four integrals which represents $f(x)$ for positive values of x; let $y \to \frac{\pi}{2} - 0$.

$\tanh(x+iy) \to (\coth x) + i0$ and $\tanh(x-iy) \to (\coth x) - i0$, or letting $\xi \equiv \coth x$ and $\xi_1 \equiv \coth x_1$ where x and x_1 are both positive, we find dropping parameters

$$\mathrm{T}(\tanh(x+iy)) \to \mathrm{T}(\xi + i0) = e^{-\frac{i\mu\pi}{2}} P(\xi) \qquad \left(P = P^{\mu}_{\nu-1/2}\right)$$

$$\mathrm{T}(\tanh(x-iy)) \to \mathrm{T}(\xi - i0) = e^{\frac{i\mu\pi}{2}} P(\xi)$$

$$\mathrm{T}(\tanh(-x-iy)) \to \mathrm{T}(-\xi - i0) = e^{\frac{i\mu\pi}{2}} P(\xi e^{-i\pi})$$

$$\mathrm{T}(\tanh(-x+iy)) \to \mathrm{T}(-\xi + i0) = e^{-\frac{i\mu\pi}{2}} P(\xi e^{i\pi})$$

Letting

$$g(\mu) \equiv \frac{\mu\pi \quad \Gamma(\frac{1}{2}+\nu-\mu)}{2\pi i \cos(\mu-\nu)\pi \, \Gamma(\frac{1}{2}+\nu+\mu)}$$ these four integrals become

$$\left.\begin{aligned}
f(x) &= \int g(\mu) P(\xi) \, d\mu \int_0^\infty f(x_1) P(\xi_1 e^{-i\pi}) \, dx_1 \\
f(x) &= \int g(\mu) P(\xi) \, d\mu \int_0^\infty f(x_1) P(\xi_1 e^{i\pi}) \, dx_1 \\
f(x) &= \int g(\mu) P(\xi e^{-i\pi}) \, d\mu \int_0^\infty f(x_1) P(\xi_1) \, dx_1 \\
f(x) &= \int g(\mu) P(\xi e^{i\pi}) \, d\mu \int_0^\infty f(x_1) P(\xi_1) \, dx_1
\end{aligned}\right\} \text{ for } 0 < x < \infty$$

From VI $(54)_a$ $(55)_a$ and $(55)_b$ we find

$$34)_a \quad \frac{1}{2}\left[P^{\mu}_{\nu-1/2}(\xi e^{i\pi}) - P^{\mu}_{\nu-1/2}(\xi e^{-i\pi})\right] = -i\cos\nu\pi\, P^{\mu}_{\nu-1/2}(\xi)$$

$$34)_b \quad \frac{1}{2}\left[P^{\mu}_{\nu-1/2}(\xi e^{i\pi}) + P^{\mu}_{\nu-1/2}(\xi e^{-i\pi})\right] = \sin\nu\pi\, P^{\mu}_{\nu-1/2}(\xi) + \frac{2\cos(\mu-\nu)\pi}{\pi\cos\mu\pi}\, Q^{\mu}_{\nu-1/2}(\xi)$$

We then obtain

$$35) \quad \frac{1}{2i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \frac{\mu\,\Gamma(\frac{1}{2}+\nu-\mu)}{\cos(\mu-\nu)\pi\,\Gamma(\frac{1}{2}+\nu+\mu)}\, P^{\mu}_{\nu-\frac{1}{2}}(\coth x)\,d\mu \int_0^\infty f(x_1)\, P^{\mu}_{\nu-\frac{1}{2}}(\coth x_1)\,dx_1 \equiv 0 \quad \text{if } 0 < x < \infty$$

This is obtained by the subtraction of the first from the second of the four integrals representing $f(x)$; and then making use of $(34)_a$.

Multiplying (35) by $-\sin\nu\pi$ and adding it to half the sum of the first two of the four integrals, gives equ $(36)_a$ below (by use of $(34)_b$). Similarly from the last two of the four we obtain $(36)_b$

$$36)_a \quad f(x) = \frac{1}{\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu\,\Gamma(\frac{1}{2}+\nu-\mu)}{\cos\mu\pi\,\Gamma(\frac{1}{2}+\nu+\mu)} P^{\mu}_{\nu-1/2}(\coth x)\,d\mu \int_0^\infty f(x_1)\,Q^{\mu}_{\nu-1/2}(\coth x_1)\,dx_1 \quad \text{where} \begin{pmatrix} -\delta < \mu_1 < \delta \\ -\nu_1 - \frac{1}{2} < \mu_1 \end{pmatrix}$$

$$36)_b \quad f(x) = \frac{1}{\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu\,\Gamma(\frac{1}{2}+\nu-\mu)}{\cos\mu\pi\,\Gamma(\frac{1}{2}+\nu+\mu)} Q^{\mu}_{\nu-1/2}(\coth x) \int_0^\infty f(x_1)\,P^{\mu}_{\nu-1/2}(\coth x_1)\,dx_1 \quad \text{where} \begin{pmatrix} -\delta < \mu_1 < \infty \\ -\nu_1 - \frac{1}{2} < \mu_1 < \infty \end{pmatrix}$$

Moving the path from $\mu_1 = \nu_1 - \frac{1}{2} + 0$ to $\mu_1 = -(\nu_1 - \frac{1}{2}) - 0$ and then letting $\mu = -\mu'$ restores the original path. Making use of the relation $\Gamma(\frac{1}{2}+\nu-\mu)\,Q^{\mu}_{\nu-1/2} = \Gamma(\frac{1}{2}+\nu+\mu)\,Q^{-\mu}_{\nu-1/2}$ converts these into

$$36)_c \quad f(x) = \frac{-1}{\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu}{\cos\mu\pi} P^{-\mu}_{\nu-1/2}(\coth x)\,d\mu \int_0^\infty f(x_1)\,Q^{\mu}_{\nu-1/2}(\coth x_1)\,dx_1 \quad \text{where} \begin{pmatrix} -\delta < \mu_1 < \delta \\ \mu_1 < \nu_1 + \frac{1}{2} \end{pmatrix}$$

$$36)_d \quad f(x) = \frac{-1}{\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu}{\cos\mu\pi} Q^{\mu}_{\nu-1/2}(\coth x)\,d\mu \int_0^\infty f(x_1)\,P^{-\mu}_{\nu-1/2}(\coth x_1)\,dx_1 \quad \text{where} \begin{pmatrix} -\infty < \mu_1 < \delta \\ -\infty < \mu_1 < \nu_1 + \frac{1}{2} \end{pmatrix}$$

Half the sum of $36)_a$ and $(36)_c$ is

$$36)_e \quad f(x) = \frac{-1}{\pi^2 i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu\sin\mu\pi\,\Gamma(\frac{1}{2}+\nu-\mu)}{\cos^2\mu\pi\,\Gamma(\frac{1}{2}+\nu+\mu)} Q^{\mu}_{\nu-1/2}(\coth x)\,d\mu \int_0^\infty f(x_1)\,Q^{\mu}_{\nu-1/2}(\coth x_1)\,dx_1 \quad \text{where} \begin{matrix} -\delta < \mu_1 < \delta \\ -\nu_1 - \frac{1}{2} < \mu_1 < \nu_1 + \frac{1}{2} \end{matrix}$$

found by use of the fundamental relation VI (6)$_a$, which is

$$Q^{\mu}_{\nu-1/2} = -\frac{\pi}{2}\cot\mu\pi\left[P^{\mu}_{\nu-1/2} - \frac{\Gamma(\frac{1}{2}+\nu+\mu)}{\Gamma(\frac{1}{2}+\nu-\mu)} P^{-\mu}_{\nu-1/2}\right].$$

The following variations may be placed here for reference. They arise by letting $\eta = \coth x$ $F(\eta) = f(x)$ where $\int_1^\infty \frac{|F(\eta)|}{\eta^2-1} d\eta$ converges and $\lim_{\eta \to 1+0} (\eta^2-1)^{\mu_1} F(\eta) = 0$ if $\mu_1 < \frac{\delta}{2}$, where δ is a given positive constant. Also $F(\eta) \sim \frac{C}{\eta^{\delta_0}}$ as $\eta \to \infty$ where $\nu_1 - 1 < \delta_0$ and $-1 < \delta_0$

If F satisfies these conditions, it is then represented for the real range $1 < \eta < \infty$ by the following which are equivalent to $(36)_{a,b,c}$ in which the constant ν has been replaced by $\nu + 1/2$ so that in these integrals $\nu = \nu_1 + i\nu_2$ where $-\frac{1}{2} \leq \nu_1$.

$$37)_a \quad F(\eta) = \frac{1}{\pi i} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu \quad \Gamma(\nu-\mu+1)}{\cos\mu\pi\,\Gamma(\nu+\mu+1)} P_\nu^\mu(\eta)\, d\mu \int_1^\infty \frac{F(\eta_1)}{\eta_1^2 - 1} Q_\nu^\mu(\eta_1)\, d\eta_1 \quad \text{where} \left(\begin{array}{l} -\delta < \mu_1 < \delta \\ -\nu_1 - 1 < \mu_1 \end{array} \right.$$

$$37)_b \quad F(\eta) = \frac{1}{\pi i} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu \;\Gamma(\nu-\mu+1)}{\cos\mu\pi\,\Gamma(\nu+\mu+1)} Q_\nu^\mu(\eta)\, d\mu \int_1^\infty \frac{F(\eta_1)}{\eta_1^2 - 1} P_\nu^\mu(\eta_1)\, d\eta_1 \quad \text{where} \left(\begin{array}{l} -\delta < \mu_1 < \infty \\ -\nu_1 - 1 < \mu_1 < \infty \end{array} \right.$$

$$37)_c \quad F(\eta) = \frac{-1}{\pi^2 i} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu \sin\mu\pi\,\Gamma(\nu-\mu+1)}{\cos^2\mu\pi\,\Gamma(\nu+\mu+1)} Q_\nu^\mu(\eta)\, d\mu \int_1^\infty \frac{F(\eta_1)}{\eta_1^2 - 1} Q_\nu^\mu(\eta_1)\, d\eta_1 \quad \text{where} \left(\begin{array}{l} -\delta < \mu_1 < \delta \\ -\nu_1 - 1 < \mu_1 < \nu_1 \end{array} \right.$$

These apply when $1 < \eta < \infty$.

By use of Whipple's transformation VI (62)d these integrals may be put in a form in which the complex integration is made with respect to the lower parameter. Eq (36) a, b, c become

$$38)_a \quad f(x)=\frac{\sqrt{\sinh x}}{i\pi\cos\nu\pi}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\mu\,\frac{\Gamma(\frac{1}{2}-\nu+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\,Q^{\nu}_{\mu-1/2}(\cosh x)\,d\mu\int_0^{\infty}f(x_1)\sqrt{\sinh x_1}\,P^{\nu}_{\mu-1/2}(\cosh x_1)\,dx_1 \quad \left(\begin{array}{l}-\delta<\mu_1<\delta\\ -\nu_1-\frac{1}{2}<\mu_1\end{array}\right.$$

$$38)_b \quad f(x)=\frac{\sqrt{\sinh x}}{i\pi\cos\nu\pi}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\mu\,\frac{\Gamma(\frac{1}{2}-\nu+\mu)}{\Gamma(\frac{1}{2}+\nu+\mu)}\,P^{\nu}_{\mu-1/2}(\cosh x)\,d\mu\int_0^{\infty}f(x_1)\sqrt{\sinh x_1}\,Q^{\nu}_{\mu-1/2}(\cosh x_1)\,dx_1 \quad \left(\begin{array}{l}-\delta<\mu_1<\infty\\ -\nu_1-\frac{1}{2}<\mu_1<\infty\end{array}\right.$$

$$38)_c \quad f(x)=\frac{-\sqrt{\sinh x}}{2i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\mu\sin\mu\pi\,\Gamma(\frac{1}{2}-\nu+\mu)}{\cos(\mu-\nu)\pi\,\Gamma(\frac{1}{2}+\nu+\mu)}\,P^{\nu}_{\mu-1/2}(\cosh x)\,d\mu\int_0^{\infty}f(x_1)\sqrt{\sinh x_1}\,P^{\nu}_{\mu-1/2}(\cosh x_1)\,dx_1$$

where $-\delta<\mu_1<\delta$ and $-\nu_1-\frac{1}{2}<\mu_1<\nu_1+\frac{1}{2}$

Or letting $\eta=\cosh x$ and $F(\eta)=(\sinh x)^{-\frac{1}{2}}f(x)$

$$39)_a \quad f(x)=\frac{1}{i\pi\cos\mu\pi}\int_{\nu_1-i\infty}^{\nu_1+i\infty}(\nu+\tfrac{1}{2})\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,Q^{\mu}_{\nu}(\eta)\,d\nu\int_1^{\infty}F(\eta_1)\,P^{\mu}_{\nu}(\eta_1)\,d\eta_1 \quad \text{where} \left(\begin{array}{l}-\delta_1-1<\nu_1<\delta_1\\ -\mu_1-1<\nu_1\end{array}\right.$$

$$39)_b \quad f(x)=\frac{1}{i\pi\cos\nu\pi}\int_{\nu_1-i\infty}^{\nu_1+i\infty}(\nu+\tfrac{1}{2})\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\,P^{\mu}_{\nu}(\eta)\,d\nu\int_1^{\infty}F(\eta_1)\,Q^{\mu}_{\nu}(\eta_1)\,d\eta_1 \quad \text{where} \left(\begin{array}{l}-\delta_1-1<\nu_1<\infty\\ -\mu_1-1<\nu_1<\infty\end{array}\right.$$

$$39)_c \quad f(x)=\frac{1}{2i}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\frac{(\nu+\frac{1}{2})\cos\nu\pi\,\Gamma(\nu-\mu+1)}{\sin(\nu-\mu)\pi\,\Gamma(\nu+\mu+1)}\,P^{\mu}_{\nu}(\eta)\,d\nu\int_1^{\infty}F(\eta_1)\,P^{\mu}_{\nu}(\eta_1)\,d\eta_1, \quad \text{where} \left(\begin{array}{l}-\delta_1-1<\nu_1<\delta_1\\ -\mu_1-1<\nu_1<\mu_1\end{array}\right.$$

where

The last three equations apply for $1<\eta<\infty$ to a function $F(\eta)$ which satisfies the conditions

$$39)_d \begin{cases} \int_1^\infty (\eta^2-1)^{-1/4} F(\eta)\, d\eta \text{ converges, } F(\eta) \sim C_1 \eta^{-\delta_1} \text{ as } \eta \to \infty \quad \text{where } \delta_1 \geq 0 \\ F(\eta) \sim C_0 (\eta^2-1)^{\delta_0} \text{ as } \eta \to 1+0 \text{ where } \frac{\mu_1}{2}-1 < \delta_0 \text{ and } -\frac{3}{2} < \delta_0 \end{cases}$$

The three equations (38) for the case $\nu = m$, may be written

$$40)_a \quad f(x) = \frac{(-1)^m \sqrt{\sinh x}}{\pi i} \int_{\nu_1 - i\infty}^{\nu_1 + i\infty} (\nu+\tfrac{1}{2}) \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} Q_\nu^m(\cosh x)\, d\nu \int_0^\infty f(x') \sqrt{\sinh x'}\, P_\nu^m(\cosh x')\, dx' \quad \begin{pmatrix} -\delta-\frac{1}{2} < \nu_1 < \delta - \frac{1}{2} \\ -m-1 < \nu_1 < m \end{pmatrix}$$

$$40)_b \quad f(x) = \frac{(-1)^m \sqrt{\sinh x}}{\pi i} \int_{\nu_1 - i\infty}^{\nu_1 + i\infty} (\nu+\tfrac{1}{2}) \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cosh x)\, d\nu \int_0^\infty f(x') \sqrt{\sinh x'}\, Q_\nu^m(\cosh x')\, dx' \quad \begin{pmatrix} -\delta-\frac{1}{2} < \nu_1 < \infty \\ -m-1 < \nu_1 < \infty \end{pmatrix}$$

$$40)_c \quad f(x) = \frac{(-1)^m \sqrt{\sinh x}}{2i} \int_{\nu_1 - i\infty}^{\nu_1 + i\infty} (\nu+\tfrac{1}{2}) \cot \nu\pi \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} P_\nu^m(\cosh x)\, d\nu \int_0^\infty f(x') \sqrt{\sinh x'}\, P_\nu^m(\cosh x')\, dx' \quad \begin{pmatrix} -\delta-\frac{1}{2} < \nu_1 < \delta - \frac{1}{2} \\ -m-1 < \nu_1 < m \end{pmatrix}$$

where in (32) $\nu_1 - 1 < \delta_0$ and $-1 < \delta_0$

In all the equations (36) to (40) the path may be taken along the imaginary axis, in which case the exponential vanishing of $f(x)$ when $x \to +\infty$ may be replaced by mere vanishing.

3. Application to Cylinder Functions.

A particular case of eq(3) is

41)$_a$ $\mathcal{D}_z^2 E(z) + \left[\frac{1/4 - \nu^2}{(z-ib)^2} - \mu^2\right] E(z) = 0$ where $b > 0$ and

ν is an arbitrary parameter, and $z = x+iy$ where $y < b$ so that as x ranges from $-\infty$ to ∞, the argument of $z-ib$ increases from $-\pi$ to zero. If $\mu \equiv \mu_1 + i\mu_2$ is any point of the μ half plane $0 < \mu_1 < \infty$, then $-\frac{\pi}{2} < \arg\mu < \frac{\pi}{2}$ and if $i\mu$ denotes $\mu e^{\frac{i\pi}{2}}$ then $0 < \arg i\mu < \pi$. Hence if ζ be defined by

41)$_b$ $\zeta \equiv i\mu(z-ib)$ then $-\pi < \arg\zeta < \pi$

Eq(41)$_a$ becomes

41)$_c$ $\mathcal{D}_\zeta^2 E + \left[1 - \frac{\nu^2 - 1/4}{\zeta^2}\right] E = 0$, that is,

41)$_d$ $\left[\mathcal{D}_\zeta^2 + \frac{1}{\zeta}\mathcal{D}_\zeta + 1 - \frac{\nu^2}{\zeta^2}\right] \zeta^{\frac{1}{2}} E = 0$ (Bessel's equation).

The function $\zeta^{\frac{1}{2}}$ and cylinder functions of ζ will be single-valued throughout the complex ζ-plane, which is cut along its negative real axis.

The relations between Bessel's and Neumann's functions $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ and the two Hankel's functions $H_1^\nu(\zeta)$ and $H_2^\nu(\zeta)$

are

42)$_a$ $$\begin{cases} H_1^\nu(\varsigma) = J_\nu(\varsigma) + iY_\nu(\varsigma) \quad \text{and} \quad H_2^\nu(\varsigma) = J_\nu(\varsigma) - iY_\nu(\varsigma) \\ \text{or} \\ 2J_\nu(\varsigma) = H_1^\nu(\varsigma) + H_2^\nu(\varsigma) \quad \text{and} \quad 2iY_\nu(\varsigma) = H_1^\nu(\varsigma) - H_2^\nu(\varsigma) \end{cases}$$

Also

42)$_b$ $$H_1^\nu(\varsigma) = \frac{i}{\sin\nu\pi}\left[e^{-i\nu\pi}J_\nu(\varsigma) - J_{-\nu}(\varsigma)\right] \text{ and } H_2^\nu(\varsigma) = \frac{-i}{\sin\nu\pi}\left[e^{i\nu\pi}J_\nu(\varsigma) - J_{-\nu}(\varsigma)\right]$$

where

42)$_c$ $$J_\nu(\varsigma) \equiv \sum_{s=0}^{\infty} \frac{(-1)^s\left(\frac{\varsigma}{2}\right)^{\nu+2s}}{\Gamma(s+1)\,\Gamma(s+\nu+1)}$$

If p is any integer the circuital relations for the branch point, $\varsigma = 0$, are

43)$_a$ $$J_\nu(\varsigma e^{ip\pi}) = e^{i\nu p\pi} J_\nu(\varsigma)$$

43)$_b$ $$Y_\nu(\varsigma e^{ip\pi}) = e^{-i\nu p\pi} Y_\nu(\varsigma) + \frac{2i\cos\nu\pi\sin p\nu\pi}{\sin\nu\pi} J_\nu(\varsigma).$$

43)$_c$ $$\sin\nu\pi\, H_1^\nu(\varsigma e^{ip\pi}) = -\sin(p-1)\nu\pi\, H_1^\nu(\varsigma) - \sin p\nu\pi\, e^{-i\nu\pi} H_2^\nu(\varsigma)$$

43)$_d$ $$\sin\nu\pi\, H_2^\nu(\varsigma e^{ip\pi}) = \sin p\nu\pi \cdot e^{i\nu\pi} H_1^\nu(\varsigma) + \sin(p+1)\nu\pi\, H_2^\nu(\varsigma)$$

Also for change of sign of the parameter ν, there are the relations

44)$_a$ $$H_1^{-\nu}(\varsigma) = e^{i\nu\pi} H_1^\nu(\varsigma) \quad = -H_2^\nu(\varsigma e^{i\pi})$$

44)$_b$ $$H_2^{-\nu}(\varsigma) = e^{-i\nu\pi} H_2^\nu(\varsigma) \quad = -H_1^\nu(\varsigma e^{i\pi})$$

It is also necessary to refer to the relations

45)$_a$ $$J_\nu(\zeta)\,Y_\nu'(\zeta) - J_\nu'(\zeta)\,Y_\nu(\zeta) = \frac{2}{\pi\zeta}$$

45)$_b$ $$H_1^\nu(\zeta)\,H_2^{\nu\prime}(\zeta) - H_1^{\nu\prime}(\zeta)\,H_2^\nu(\zeta) = \frac{4}{\pi i\zeta}$$

and the asymptotic expansions

46)$_a$ $$\sqrt{\frac{\pi\zeta}{2}}\,H_1^\nu(\zeta) \sim e^{i[\zeta-(\nu+\frac{1}{2})\frac{\pi}{2}]}$$

46)$_b$ $$\sqrt{\frac{\pi\zeta}{2}}\,H_2^\nu(\zeta) \sim e^{-i[\zeta-(\nu+\frac{1}{2})\frac{\pi}{2}]}$$

$$|\zeta| \to \infty, \quad -\pi < \arg\zeta < \pi$$

In the contour integrals below, the cut in the ζ-plane is a barrier, across which the path will not be deformed, so that the application of the circuital relations (43) will be limited to cases where the points ζ and $\zeta e^{ip\pi}$ both lie in the cut, ζ-plane. This limits the values of the integer p to ± 1 in general, and to ± 2 when the points are adjacent on opposite sides of the cut.

Two fundamental solutions of (41)$_a$ are

47)$_a$ $$E_1^\mu(z) \equiv \sqrt{\frac{\pi\zeta}{2}}\,e^{-i[\mu b-(\nu+\frac{1}{2})\frac{\pi}{2}]}\,H_1^\nu(\zeta)$$

47)$_b$ $$E_2^\mu(z) \equiv \sqrt{\frac{\pi\zeta}{2}}\,e^{i[\mu b-(\nu+\frac{1}{2})\frac{\pi}{2}]}\,H_2^\nu(\zeta)$$

When $|z| \to \infty$, or when $|\mu| \to \infty$ the definition $(41)_e$ shows that $|\xi| \to \infty$, so that by $(46)_a$ these definitions of $E_1^\mu(z)$ and $E_2^\mu(z)$ make

48) $E_1^\mu(z) \to e^{-\mu z}$ and $E_2^\mu(z) \to e^{\mu z}$ when $|z| \to \infty$ and when $|\mu| \to \infty$

so the requirements A, B, C, D, etc, of the general formulation are satisfied. Also from $(45)_e$ we find

49) $\Omega(0,\mu,\mu) = 2\mu = E_1^\mu(z)\, E_2^{\mu\prime}(z) - E_1^{\mu\prime}(z)\, E_2^\mu(z)$

Hence the integral representation (6) becomes, for $-\infty < x < \infty$,

$$50)_a \quad f(x) = \frac{\sqrt{x+iy-ib}}{4}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \mu\, H_1^\nu(i\mu(x+iy-ib))\,d\mu \int_{-\infty}^{\infty} f(x_1)\sqrt{x_1+iy-ib}\; H_2^\nu(i\mu(x_1+iy-ib))\,dx_1$$

$$50)_b \quad f(x) = \frac{\sqrt{x+iy-ib}}{4}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \mu\, H_2^\nu(i\mu(x+iy-ib))\,d\mu \int_{-\infty}^{\infty} f(x_1)\sqrt{x_1+iy-ib}\; H_1^\nu(i\mu(x_1+iy-ib))\,dx_1$$

where $0 < \mu_1 < \delta$

For the positive range only, place $f(x) \equiv 0$ for $-\infty < x < 0$ and consider the limiting case $y \to b-0$. For this let $f(x) = \sqrt{x}\,F(x)$ when $x > 0$ where $\int_0^\infty \sqrt{x}\,|F(x)|\,dx$ and $\int_0^\infty x\,F(x)\,H_1^\nu(\mu x)\,dx$ converge when $\mu_1 > 0$

The equation $(50)_b$ becomes after making the substitution $\mu' = i\mu$

51) $$F(x) = \frac{1}{4}\int_{-\infty+i0}^{\infty} \mu H_2^{\nu}(\mu x)\,d\mu \int_0^{\infty} x_1 F(x_1) H_1^{\nu}(\mu x_1)\,dx_1 \quad \text{for} \quad 0 < x < \infty$$

where this μ-plane is cut along the negative real axis and the first half of the path is just above the cut.

This may be brought into other forms by noting that

52) $$0 = \frac{1}{4}\int_{-\infty+i0}^{\infty} \mu\, A(\mu x)\,d\mu \int_0^{\infty} x_1 F(x_1) B(\mu x_1)\,dx_1 = 0 \quad \text{when} \quad 0 < x < \infty$$

where $A(z)$ and $B(z)$ are analytic functions of z in the upper half-plane, one of which, say $B(z)$ has the asymptotic expansion, when $|z| \to \infty$, $\quad 0 < \arg z < \pi$

$B(z) \sim e^{iz}$ times a finite number of powers of z (positive, or negative, and not necessarily integral).

The other function $A(z)$ may have the same type of asymptotic expansion, or the exponential factor e^{iz} may be absent. Then (52) is equivalent to the same integral taken along an infinite semicircle; which is zero.

Taking $B(z) = H_1^{\nu}(z)$, adding (52) to (51) and then making the substitution $\mu = \mu' e^{i\pi}$ in that part of the integral which is taken along the cut, gives after making use of $(43)_2$ for $p=1$,

$$53)\quad F(x) = \frac{1}{4}\int_0^\infty \mu\, d\mu \int_0^\infty dx_1 x_1 F(x_1) \left\{ \left[A(\mu x) + H_2^\nu(\mu x) \right] H_1^\nu(\mu x_1) + e^{-i\nu\pi} \left[A(\mu x e^{i\pi}) + H_2^\nu(\mu x e^{i\pi}) \right] H_2^\nu(\mu x_1) \right\}$$

A generalization of cylinder functions is Lommel's function with two parameters

$$54)_a\quad \Pi^{\nu,\sigma}(z) \equiv \cos(\sigma-\nu)\frac{\pi}{2} \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{2}\right)^{\sigma+2s}}{\Gamma\left(s+1+\frac{\sigma+\nu}{2}\right)\Gamma\left(s+1+\frac{\sigma-\nu}{2}\right)}$$

This has the asymptotic expansion when $|z| \to \infty$, valid in the z-plane cut along its negative real axis,

$$54)_b\quad 2\Pi^{\nu,\sigma}(z) \sim \cos(\sigma-\nu)\frac{\pi}{2} \left[e^{-(\sigma-\nu)\frac{i\pi}{2}} H_1^\nu(z) + e^{(\sigma-\nu)\frac{i\pi}{2}} H_2^\nu(z) + z^\sigma S(z) \right]$$ where

$S(z)$ consists of a finite number of positive and negative integral powers of z^2 (N. Nielsen: Handbuch d. Theorie der Cylinderfunktionen p 228).

Hence $A(z)$ will have the required type of expansion if we take

$$A(z) = C\left[2\Pi^{\nu,\sigma}(z) - e^{(\sigma-\nu)\frac{i\pi}{2}} H_2^\nu(z) \right]$$

Taking the constant $C = \dfrac{e^{-(\sigma-\nu)\frac{i\pi}{2}}}{\cos(\sigma-\nu)\frac{\pi}{2}}$, eq (53) becomes

$$55)\quad F(x) = \frac{e^{-(\sigma+\nu)\frac{i\pi}{2}}}{2\cos(\sigma-\nu)\frac{\pi}{2}} \int_0^\infty \mu \Pi^{\nu,\sigma}(\mu x)\, d\mu \int_0^\infty x_1 F(x_1) \left[e^{i\nu\pi} H_1^{\nu}(\mu x_1) + e^{i\sigma\pi} H_2^{\nu}(\mu x_1) \right] dx_1$$

In the special case $\sigma = \nu$, $\Pi^{\nu,\nu}(z) = J_\nu(z)$ and this becomes Hankel's integral identity

$$56)\quad F(x) = \int_0^\infty \mu J_\nu(\mu x)\, d\mu \int_0^\infty x_1 F(x_1)\, J_\nu(\mu x_1)\, dx_1$$

Also writing $\dfrac{\Pi^{\nu,\sigma}(\mu x)}{\cos(\sigma-\nu)\frac{\pi}{2}} = \dfrac{\Pi^{-\nu,\sigma}(\mu x)}{\cos(\sigma+\nu)\frac{\pi}{2}}$ by $(54)_a$

and then taking $\sigma = \nu + 1$ gives

$$57)\quad F(x) = \int_0^\infty \mu Z_\nu(\mu x)\, d\mu \int_0^\infty x_1 F(x_1)\, Y_\nu(\mu x_1)\, dx_1 = \int_0^\infty \mu Y_\nu(\mu x)\, d\mu \int_0^\infty x_1 F(x_1)\, Z_\nu(\mu x_1)\, dx_1$$

where

$$57)_a\quad Z_\nu(z) \equiv \frac{\Pi^{-\nu,\nu+1}(z)}{-\sin\nu\pi} \equiv \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{2}\right)^{2s+\nu+1}}{\Gamma(s+\frac{3}{2})\Gamma(s+\frac{3}{2}+\nu)} = \text{Struve's function.}$$

Also multiplying (55) by $e^{(\sigma+\nu)\frac{i\pi}{2}} \cos(\sigma-\nu)\frac{\pi}{2}$, differentiating the equation with respect to σ and then placing $\sigma = \nu$, gives

$$58)\quad F(x) = \int_0^\infty \mu\, d\mu \int_0^\infty x_1 F(x_1) \left[J_\nu(\mu x)\, H_2^{\nu}(\mu x_1) - \frac{i}{\pi} L_\nu(\mu x)\, J_\nu(\mu x_1) \right] dx_1$$

where

$$59)\quad L_\nu(z) \equiv 2\left[D_\sigma \Pi^{\nu,\sigma}(z) \right]_{\sigma=\nu} = 2 J_\nu(z) \log\frac{z}{2} - \sum_{s=0}^{\infty} \frac{(-1)^s \left(\frac{z}{2}\right)^{\nu+2s}}{\Gamma(s+1)\Gamma(s+1+\nu)} \left[\psi(s+1) + \psi(s+1+\nu) \right]$$

The integral cosine $C_i(z)$ and integral sine $S_i(z)$ are expressible in terms of $L_{1/2}(z)$ and $L_{-1/2}(z)$ for

60)$_a$ $$L_{\frac{1}{2}}(z) = 2\sqrt{\frac{2}{\pi z}}\left[\sin z\, C_i(z) - \cos z\, S_i(z)\right]$$

60)$_b$ $$L_{-\frac{1}{2}}(z) = 2\sqrt{\frac{2}{\pi z}}\left[\cos z\, C_i(z) + \sin z\, S_i(z)\right]$$

In the case $\nu = -\frac{1}{2}$ the real part of (58) becomes Fourier's cosine integral for $F(x)$ and in the case $\nu = \frac{1}{2}$ it becomes his sine integral.

IX Some integral equations of potential theory with $Q_{m-1/2}$ as nucleus. Its canonical expansions.

1 The reduced potential.

If a potential V has its boundary values given on a surface of revolution it is convenient for the general formulation to use circular cylindrical coordinates (x, ρ, ϕ). Since V and all of its derivatives are periodic functions of ϕ with period 2π it may be expanded in a differentiable Fourier's series

1) $$V(x,\rho,\phi) = C_0 + C_1 x + (C_2 + C_3 x)\log\rho + \sum_{m=0}^{\infty} V^m(x,\rho)\cos m(\phi - \phi_m)$$

Where there are no sources, V satisfies Laplace's equation

2) $$\nabla^2 V \equiv \left(D_x^2 + D_\rho^2 + \frac{1}{\rho}D_\rho + \frac{1}{\rho^2}D_\phi^2\right)V(x,\rho,\phi) = 0.$$

Each coefficient $V^m(x,\rho)$ must be a solution of the equation

3) $$\left(D_x^2 + D_\rho^2 + \frac{1}{\rho}D_\rho - \frac{m^2}{\rho^2}\right)V^m(x,\rho) = 0.$$

If we let

4) $V^m(x,\rho) = \rho^{-\frac{1}{2}} U^m(x,\rho)$ the "reduced" potential $U^m(x\rho)$

is a solution of

5) $\left[D_x^2 + D_\rho^2 + \frac{1/4 - m^2}{\rho^2}\right] U^m(x,\rho) = 0$ which in a slightly different form was called Euler's equation by Darboux.

Its particular solutions of the form $U^m = A_m \rho^{\frac{1}{2}+m} + A_{-m}\rho^{\frac{1}{2}-m}$ correspond to a potential independent of x, of the form $V(\rho,\phi) = \sum_{m=-\infty}^{\infty} A_m \rho^m \cos m(\phi-\phi_m)$.

The Newtonian potential $V_c(x,\rho,\phi)$ at any point (x,ρ,ϕ) due to a circular line charge in the plane x_1 of radius ρ_1, coaxial with the x axis, and with linear density $\cos m(\phi-\phi_m)$ is

$V_c(x,\rho,\phi) = \rho_1 \int_{-\pi}^{\pi} \frac{\cos m(\phi_1-\phi_m)}{R} d\phi_1$ where R is the distance from the point (x,ρ,ϕ) to the point of integration (x_1,ρ_1,ϕ_1). In the half plane $0<\rho<\infty$, $-\infty<x<\infty$ the trace of this circle appears as a singular point or source at $(x_1\rho_1)$ for the reduced potential. If D denotes the distance, measured in a meridian plane ϕ=constant, from (x,ρ) to (x_1,ρ_1) then $D^2 = (x-x_1)^2 + (\rho-\rho_1)^2$ and

$$\frac{1}{R} = \frac{1}{\sqrt{2\rho\rho_1}}\left[1 + \frac{D^2}{2\rho\rho_1} - \cos(\phi-\phi_1)\right]^{-\frac{1}{2}}$$

Hence by VI $(73)_8$ if we place $\mu = 0$ and $z = 1 + \frac{D^2}{2\rho\rho_1}$

we obtain the expansion

6) $$\frac{1}{R} = \frac{2}{\pi\sqrt{\rho\rho_1}} \sum_{m=0}^{\infty} \epsilon_m Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right) \cos m(\phi-\phi_1) \qquad \begin{array}{l} \epsilon_0 = \frac{1}{2} \\ \epsilon_m = 1 \text{ if } m \neq 0 \end{array}$$

The potential becomes, by use of this,

$V_c(x,\rho,\phi) = \frac{U_c^m(x,\rho)}{\sqrt{\rho}} \cos m(\phi-\phi_m)$ where the reduced potential is

$U_c^m(x,\rho) = 2\sqrt{\rho_1}\, Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)$ which is a solution of Euler's two-dimensional equation (5) with a singular point at (x_1,ρ_1). Since we have called U^m a (reduced) potential we may borrow other terms from ordinary potential theory, such as (reduced) charge and (reduced) density of charge. The advantage of this loan will soon appear.

Let s denote a curve in the $x\rho$ half-plane. We may speak of the point s or the point $\dot{s}$, meaning points on the curve. Suppose that upon the surface of revolution S whose trace is the curve s, there is a simple distribution of electric charge with surface density $\sigma(s)\cos m(\phi-\phi_m)$. Then its Newtonian potential is

$$V(x,\rho,\phi) = \frac{U^m(x,\rho)}{\sqrt{\rho}} \cos m(\phi-\phi_m)$$

where the reduced potential $U^m(x,\rho)$ is given at any

point (x, ρ) in the $x\rho$ half plane by the line integral

7) $$\overset{m}{U}(x,\rho) = 2\int_0^l \bar{\sigma}(s_1)\, Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)ds_1$$ where x_1, ρ_1 (or s_1) is the point of integration on the curve and the reduced density is defined by

8) $$\bar{\sigma}(s_1) = \sqrt{\rho_1}\,\sigma(s_1)$$

The potential $\overset{m}{U}(x,\rho)$ is analogous to the logarithmic potential of a simple distribution with density $\bar{\sigma}(s_1)$ on an endless cylinder whose trace in the $x\rho$ half plane is the "charged curve" s. In that case the function $Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)$ is replaced by $-\log D$. It will be seen by $(12)_2$ below that when $(x,\rho) \to (x_1, \rho_1)$, i.e. when $D \to 0$, $Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right)$ becomes infinite in the same manner as $-\log D$, that is

9) $$Q_{m-1/2}\left(1+\frac{D^2}{2\rho\rho_1}\right) \sim -\log D$$ plus terms which remain finite when $D=0$.

It is due to this fact primarily that the analogy between reduced potential and logarithmic potential is so far reaching — their contrasts will also appear. As to similar features it is obvious that $\overset{m}{U}$ is continuous at the charged curve but its normal derivatives have discontinuities there which

are connected with the (reduced) density $\bar{\sigma}(s)$ of the simple distribution at the point, by the classical relation

10)$_a$ $4\pi\bar{\sigma}(s) = -D_n U^m_{(s+0)} + D_n U^m_{(s-0)}$

where the meaning of this notation is as follows. If the curve s is a simple closed curve in the $x\rho$ half plane, which does not touch the x axis the meaning of "inside" and "outside" of the curve is apparent. The same is true of an area bounded externally by such a curve and bounded internally by such a one. Also if the curve begins and ends on the x axis the "inside" and "outside" are evident. The normal n may be understood as always having the direction of the exterior normal. If s and s_1 are two points on the curve $D_n U$ and $D_{n_1} U$ denote derivatives in the direction of this exterior normal with respect to the coordinates (x, ρ) or (x_1, ρ_1) which approach s and s_1 respectively. To distinguish between the values of these derivatives when taken just outside or just inside the curve we use the notation $D_n U(s+0)$ and $D_n U(s-0)$ respectively, the direction n in all cases being the same — that of exterior normal. If the curve is not closed in one of the above senses, then

either side may be taken arbitrarily as the outside.

If we differentiate the integral (7) in a fixed direction n, with respect to the coordinates (x,ρ) and then let (x,ρ) approach a point s on the curve (where the normal n is that fixed direction chosen beforehand) we find when the approach is from within

$$10)_b \qquad \mathfrak{D}_n U^m_{(s-0)} = 2\pi\bar{\sigma}(s) + 2\int_0^\ell \bar{\sigma}(s_1)\,\mathfrak{D}_n Q_{m-1/2}(g(s,s_1))\,ds_1$$

When the approach is from the outside we obtain

$$10)_c \qquad \mathfrak{D}_n U^m_{(s+0)} = -2\pi\bar{\sigma}(s) + 2\int_0^\ell \bar{\sigma}(s_1)\,\mathfrak{D}_n Q_{m-1/2}(g(s,s_1))\,ds_1$$

Where $g(s,s_1)$ is defined by (11) below when $(x,\rho)\to s$ and $(x_1\rho_1)\to s_1$ and $\mathfrak{D}_n Q_{m-\frac{1}{2}}(g(s,s_1))$ means $Q'_{m-\frac{1}{2}}(g(s,s_1))\cdot\left[\mathfrak{D}_n\, g(x,\rho;x_1\rho_1)\right]_{x,\rho\to s}$ the differentiation being with respect to $(x\rho)$. This is in general continuous, except when $s\to s_1$ but the integrals converge.

To examine the character at the boundary ($\rho=0$, or $\sqrt{x^2+\rho^2}\to\infty$) of potentials U^m which are defined by line integrals like (7) let $\bar{D}$ denote the distance from the point $x\rho$ in the half plane to the point $(x_1,-\rho_1)$ so that $\bar{D}^2 = (x-x_1)^2+(\rho+\rho_1)^2 = D^2+4\rho\rho_1$

Let

$$11) \quad q(x,\rho;x_1,\rho_1) = q(x,\rho) = 1 + \frac{D^2}{2\rho\rho_1} = \frac{\bar{D}^2 + D^2}{\bar{D}^2 - D^2}$$

By VI (43) we find that when (x,ρ) is any point in the half plane

$$12)_a \quad Q_{m-1/2}(q) = Q_{-m-1/2}(q) = \left(\frac{\rho\rho_1}{\bar{D}^2}\right)^{m+\frac{1}{2}} \frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!} F\left(m+\tfrac{1}{2}, m+\tfrac{1}{2}, 2m+1; \frac{4\rho\rho_1}{\bar{D}^2}\right)$$

Also by VI (46)

$$12)_b \quad Q_{m-1/2}(q) = P_{m-1/2}(q) \log\frac{\bar{D}}{D} + \left(\frac{\bar{D}^2}{4\rho\rho_1}\right)^{m-\frac{1}{2}} \sum_{s=0}^{\infty} \left(\frac{D}{\bar{D}}\right)^{2s} \frac{[\psi(s+1) - \psi(s-m+\frac{1}{2})]}{\Gamma^2(s+1)\,\Gamma^2(m+\frac{1}{2}-s)}$$

where by VI (45)

$$12)_c \quad P_{m-1/2}(q) = \left(\frac{4\rho\rho_1}{\bar{D}}\right)^{m+\frac{1}{2}} F\left(m+\tfrac{1}{2}, m+\tfrac{1}{2}, 1; \frac{D^2}{\bar{D}^2}\right)$$

which shows that $P_{m-1}(q) \to 1$ when $D \to 0$ (since $\bar{D}^2 \to 4\rho\rho_1$)

At the infinite semicircle $\sqrt{x^2+\rho^2} \to \infty$, $D \to \infty$, $\bar{D} \to \infty$ and $1 - \frac{D^2}{\bar{D}^2} \to 0$ so by $(12)_a$, $Q_{m-1/2}(q)$ vanishes for if $\frac{\rho}{D} = \sin\theta$

$$12)_d \quad Q_{m-1/2}(q) \sim \left(\frac{\rho\rho_1}{D^2}\right)^{m+\frac{1}{2}} \frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!} = \left[\frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})\sin\theta^{m+\frac{1}{2}}}{m!}\right] \frac{\rho_1^{m+\frac{1}{2}}}{D^{m+\frac{1}{2}}} \quad (\quad \to \infty \quad)$$

Also when $\rho \to 0$ $\quad Q_{m-\frac{1}{2}}(q)$ vanishes like $\rho^{m+\frac{1}{2}}$, for $(12)_a$ gives

$$12)_e \quad Q_{m-1/2}(q) \sim \frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!}\left(\frac{\rho\rho_1}{D_0^2}\right)^{m+\frac{1}{2}} \left[1 - (m+\tfrac{1}{2})\frac{\rho^2}{D_0^2}\left(1 - \frac{m+3/2}{m+1}\frac{\rho_1^2}{D_0^2}\right) + \text{Zero}\,\frac{\rho^4}{D_0^4}\right]$$

$$\sim \left[\frac{\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!}\right] \rho^{m+\frac{1}{2}} \cdot \left(\frac{\sin\theta_1}{D_0}\right)^{m+\frac{1}{2}} \text{ where } \sin\theta_1 \equiv \frac{\rho_1}{D_0} \text{ and } D_0^2 \equiv (x-x_1)^2 + \rho_1^2$$

From the system of equations (12) we find the following boundary character for a function U^m defined by a line integral (7), provided that the total (reduced) charge is finite, i.e. the integral $\int_0^\ell \sigma(s_1)ds_1$ converges.

$$13)_a \quad U^m(x,\rho) \sim M_m \left(\frac{\sin\theta}{R}\right)^{m+\frac{1}{2}}\left[1 + \text{Zero}\ \frac{\rho^2}{R^4}\right] = \text{Zero}\left(\frac{1}{R^{m+\frac{1}{2}}}\right) \text{ when } R \to \infty$$

where $M_m = \frac{2\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!}\int_0^\ell \rho_1^{m+\frac{1}{2}}\sigma(s_1)ds_1$ and $R^2 = x^2+\rho^2$

$$13)_b \quad U^m(x,\rho) \sim \rho^{m+\frac{1}{2}}\left[A_m(x) + B_m(x)\rho^2\right] = \text{Zero}(\rho^{m+\frac{1}{2}}) \quad \text{when } \rho \to 0$$

this being the condition that there is no source on the x axis. From this it follows that

$$13)_b' \quad \left[U^m D_\rho U^m\right]_{\rho=0} = (m+\tfrac{1}{2})\left(\frac{U^m}{\sqrt{\rho}}\right)^2_{\rho=0} = (m+\tfrac{1}{2})\left[\rho^{2m}A_m^2(x)\right]_{\rho=0} = 0 \text{ if } m \neq 0$$

$$= \tfrac{1}{2}A_0^2(x) \text{ if } m = 0$$

If $U(x,\rho)$ is any solution of Euler's eq (5) we find

$$D_x(UD_xU) + D_\rho(UD_\rho U) = (m^2-\tfrac{1}{4})\frac{U^2}{\rho^2} + (D_xU)^2 + (D_\rho U)^2$$

$$= \left(\frac{mU}{\rho}\right)^2 + (D_xU)^2 + \rho\left[D_x\frac{U}{\sqrt{\rho}}\right]^2 + \frac{1}{2}D_\rho\left(\frac{U^2}{\rho}\right)$$

Hence, since U^m satisfies the boundary conditions (13) when

[illegible] is defined by the [illegible] as [illegible]

$$[illegible] = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{0}^{\infty} d\rho \left[[illegible] \right] [illegible]$$

the boundary relations [illegible] this [illegible] that the line integral on the [illegible] is [illegible].

From the definition of g we find that [illegible] satisfies a partial differential equation of [illegible] where $1/4 - m^2$ is replaced by -2.

$$(D_x^2 + D_\rho^2)\, g = \frac{2g}{\rho^2}$$

$$(D_x g)^2 + (D_\rho g)^2 = \frac{g^2 - 1}{\rho^2}$$

These, together with

$$(g^2-1)\, Q''_{m-1/2}(g) + 2g\, Q'_{m-1/2}(g) + (\tfrac{1}{4} - m^2)\, Q_{m-1/2}(g) = 0 ,$$

are sufficient to show that $Q_{m-1/2}(g)$ is a solution of [illegible] equation in the variables (x, ρ) and also the [illegible]. Since $P_{m-1/2}(g)$ is a solution of [illegible] it is a solution of [illegible] in either pair of variables, which is regular [illegible] but it has singularities on the x axis as shown by the [illegible]

$$\pi P_{m-1/2}(g) = 2\, Q_{m-1/2}(g) \log \frac{D^2}{4\rho\rho_0} + \left(\frac{4\rho\rho_0}{D^2}\right)^{[illegible]} \sum [illegible] \left(\frac{4\rho\rho_0}{D^2}\right)^{s} \frac{(s-1)!\, \Gamma [illegible]}{(2m-s)!} [illegible]$$

$$+ \sum_{s=0}^{\infty} \left(\frac{4\rho\rho_0}{D^2}\right)^{s+m+1/2} \frac{\Gamma^2(s+m+\tfrac{1}{2})}{s!\,(2m+s)!} \left[[illegible] \right]$$

This shows that $P_{m-1/2}(g)$ becomes infinite when either point comes on the x axis.

The function $Q_{m-1/2}(g)$ is a symmetrical function of the two points (x,ρ) and (x_1,ρ_1) both considered in the half-plane. It may be interpreted as the (reduced) potential at one of the points when there is a unit (reduced) source at the other. The analogy between reduced potential and logarithmic, survives even when integrals like (7) do not converge, its more general form then lies in the continuity of potential at the charged curve, and in the definition of σ by $(10)_a$ in terms of the discontinuity of the normal derivatives.

A contrast appears in the condition for no sources on the x axis or at infinity. Consider a function $U^m(x,\rho)$ which might be supposed to become infinite on the x axis and (or) at infinity in the following manner

$15)_a$ $U^m(x,\rho) \sim \rho^{\delta_0} C(x)$ when $\rho \to 0$ where $\int_{-\infty}^{\infty} \frac{C(x)\,dx}{(x^2+c^2)^{m+\frac{1}{2}}}$ converges.

$15)_b$ $U^m(x,\rho) \sim R^{\delta} C_0(\theta)$ when $R \to \infty$ in any direction, θ, where C_0 is bounded and $R^2 = x^2+\rho^2$ and $\sin\theta = \frac{x}{R}$.

If it be assumed that

$15)_c$ $-(m-\frac{1}{2}) < \delta_0$ and $\delta < m+\frac{1}{2}$ then $U^m(x,\rho)$ cannot

be a solution of Euler's equation throughout the $x\rho$ half-plane, for this would require $U^m(x,\rho) \equiv 0$ everywhere. The proof could be made as in $(13)_c$ where there is no charged curve, so the double integral, which converges by reason of the conditions (15), must be zero; hence $U^m \equiv 0$.

This is a special case of the following.

Assume that $U^m(x,\rho)$ while satisfying the conditions (15) is also a solution of (5) throughout the half plane except at some curve s which may extend to infinity. At this curve U^m and its normal derivatives have discontinuities which are finite in general, with isolated, integrable, infinities. Since the functions $U^m(x,\rho)$ and $Q_{m-1/2}(\xi)$ satisfy (5) in the variables (x_1,ρ_1) we find

$$\int\left[U^m D_{n_1} Q_{m-\frac{1}{2}} - Q_{m-\frac{1}{2}} D_{n_1} U^m\right] ds = 0,$$ the integral being taken around a complete boundary consisting of

a) the entire x axis

b) both sides of the curve s

c) an infinite semi-circle

d) an infinitesimal circle around the fixed point (x,ρ)

The contribution of the x axis is zero by the hypothetical condition $(15)_a$ with $-(m-\frac{1}{2}) < \delta_0$. That of the

infinite semicircle is zero by 15)$_c$ with $\delta < m+\frac{1}{2}$.
The contribution of the infinitesimal circle is $2\pi \overset{m}{U}(x,\rho)$ and that of the two sides of s may be expressed in terms of the density $\bar{\sigma}(s)$ of a simple distribution and that $\bar{\tau}(s)$ of a double distribution defined by

$$15)_d \quad 4\pi\bar{\sigma}(s) \equiv -\mathcal{D}_n \overset{m}{U}(s+0) + \mathcal{D}_n \overset{m}{U}(s-0)$$

$$15)_e \quad 4\pi\bar{\tau}(s) \equiv \overset{m}{U}(s+0) - \overset{m}{U}(s-0)$$

The result is that $\overset{m}{U}(x,\rho)$ is given at every finite point (x,ρ) of the half plane by the absolutely convergent integrals

$$15)_f \quad \overset{m}{U}(x,\rho) = 2\int_0^l \bar{\sigma}(s_1) \underset{m-1/2}{Q}(q(x,\rho;x_1,\rho_1))\,ds_1 + 2\int_0^l \bar{\tau}(s_1)\mathcal{D}_{n_1} \underset{m-1/2}{Q}(q(x,\rho;x_1,\rho_1))\,ds_1$$

Since double distributions will not be considered further, this becomes

$$15)_g \quad \overset{m}{U}(x,\sigma) = 2\int_0^l \bar{\sigma}(s_1) \underset{m-1/2}{Q}(q(x,\rho;x_1,\rho_1))\,ds_1$$

This shows that in case the curve extends to infinity $(l=\infty)$ there is a sharp contrast with logarithmic potential, for this potential integral

may converge at any finite point (x, ρ) even though the total charge is infinite, or when the density itself becomes infinite like $\bar{\sigma}(s) \sim C D^{\delta-1}$ as $D \to \infty$ provided that $\delta < m + \frac{1}{2}$ as in $(15)_e$. This is evident since $Q_{m-1/2}(q)$ vanishes like $\left(\frac{\rho \rho_1}{D^2}\right)^{m+\frac{1}{2}} = \rho^{m+\frac{1}{2}} \left(\frac{\sin\theta_1}{D}\right)^{m+\frac{1}{2}}$

This potential would satisfy the condition for no charge on the x axis, vanishing with ρ like $\rho^{m+\frac{1}{2}}$ but would become infinite when $R \to \infty$.

In case l is finite, the hypothesis that each isolated infinity of $\bar{\sigma}$ is integrable, makes the total charge finite so the integral $\int_0^l \bar{\sigma}(s_1)\, ds_1$ converges. In that case we find by $(12)_d$ that when $R \equiv \sqrt{x^2+\rho^2} \to \infty$ in the direction θ,

$$U^m(x,\rho) \to \left[\frac{2\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!}\right]\left(\frac{\sin\theta}{R}\right)^{m+\frac{1}{2}} \int_0^l \rho_1^{m+\frac{1}{2}} \bar{\sigma}(s_1)\, ds_1$$

When $\rho \to 0$ we find by $(12)_e$

$$U^m(x,\rho) \to \left[\frac{2\sqrt{\pi}\,\Gamma(m+\frac{1}{2})}{m!}\right] \rho^{m+\frac{1}{2}} \int_0^l \left[\frac{\rho_1}{(x-x_1)^2+\rho_1^2}\right]^{m+\frac{1}{2}} \bar{\sigma}(s_1)\, ds.$$

Since these integrals converge, the assumed inequalities $(15)_e$ are replaced by the equalities

$$\delta = \delta_0 = m + 1/2$$

2. Integral Equation with Nucleus $Q_{m-1/2}$

(a). General Formulation.

The problem of finding a potential $V(x,\rho,\phi)$ at every point (x,ρ,ϕ) in space, whose only sources are in the form of a simple distribution on a certain surface of revolution, on which V is given, requires the determination of each reduced potential coefficient $\overset{m}{U}(x,\rho)$ at all points in the half plane, whose values $\overset{m}{U}(s)$ are assigned on the generating curve. The integral equation to determine the reduced density $\bar{\sigma}(s)$ is by (7)

16) $$\overset{m}{U}(s) = 2\int_0^{\ell} \bar{\sigma}(s_1)\, Q_{m-1/2}(q(s,s_1))\, ds_1$$

Instead of the orthogonal pair of coordinates (x,ρ) we may use as coordinates the orthogonal pair (α,β) where α and β are conjugate functions of x and ρ defined by some equation of the form

17) $z = f(u)$ where $z \equiv x + i\rho$ and $u \equiv \alpha + i\beta$.

which maps the x,ρ half plane upon some simply-connected area of the u-plane.

If $h(\alpha,\beta)$ is the positive real defined by

18) $\dfrac{1}{h^2} \equiv \left|\dfrac{dz}{du}\right|^2 = |f'(u)|^2 = (D_\alpha x)^2 + (D_\beta x)^2 = (D_\alpha \rho)^2 + (D_\beta \rho)^2$

then

19) $(D_x^2 + D_\rho^2)\,V = h^2\,(D_\alpha^2 + D_\beta^2)\,V.$

The transform of Euler's equation (5) is

20) $\left[D_\alpha^2 + D_\beta^2 + (1/4 - m^2)\,S(\alpha,\beta)\right] U^m(\alpha,\beta) = 0$

where

21) $S(\alpha,\beta) \equiv \dfrac{1}{\rho^2 h^2} = \dfrac{1}{\rho^2}\left[(D_\alpha \rho)^2 + (D_\beta \rho)^2\right] = -(D_\alpha^2 + D_\beta^2)\log\rho.$

Suppose that the charged curve is the entire locus of the equation $\beta(x,\rho) = \beta_1 =$ constant, so that, as the arc length s increases from zero to l, the coordinate α runs through its entire range say from α_0 to α_l. The line elements ds_1 and dn_1 will be related like dx and $d\rho$ so that if we write $h(\alpha_1)$ for $h(\alpha,\beta_1)$ for brevity

22) $ds_1 = \dfrac{d\alpha_1}{h(\alpha_1)}$ and $dn_1 = \dfrac{d\beta_1}{h(\alpha_1)} = -dn_2$

The equation (7) may then be written

23) $U^m(\alpha,\beta) = 2\displaystyle\int_{\alpha_0}^{\alpha_l} \frac{\mathcal{F}(\alpha_1)}{h(\alpha_1)}\, Q_{m-1/2}\left(q(\alpha,\beta;\alpha_1,\beta_1)\right) d\alpha_1$

When $\bar{\sigma}$ is known this gives $\overset{m}{U}(\alpha,\beta)$ at all points (α,β) corresponding to all points of the $x\rho$ half plane. For brevity let $\overset{m}{U}(\alpha) = \overset{m}{U}(\alpha,\beta_1) = \overset{m}{U}(\alpha,\beta_1 \pm 0)$. Also let $g(\alpha,\alpha_1)$ denote $g(\alpha,\beta_1;\alpha_1,\beta_1)$ where $g(\alpha,\beta;\alpha_1,\beta)$ has been used to denote the transform of $g(x,\rho;x_1,\rho_1)$.

The integral equation (16) becomes

$$24)\qquad \overset{m}{U}(\alpha) = 2\int \frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1)} Q_{m-1/2}\big(g(\alpha,\alpha_1)\big)\, d\alpha_1$$

From here on, attention is confined to the case in which the charged curve $\beta=\beta_1$ in the $x\rho$ half plane is either a closed curve not touching the x axis as in fig 1_a, or it begins and ends on the x axis as in fig 1_b or fig 1_c. The case of an open curve may be obtained as a limiting case of fig $1a$ in which $\beta_1 \to \beta_0$ and the curve shrinks to both sides of the cut $\bar{AO} + \bar{OA_1}$.

If, as we assume, the $x\rho$ half plane is represented upon some simply connected area of the u-plane there will in general be such a cut as $\bar{AO}$, $\bar{A_1O}$ in fig 1_a and this cut must be continued to the boundary of the $x\rho$ half plane (either the x axis or the infinite semi-circle. This continuation of the cut (to the x axis) is shown

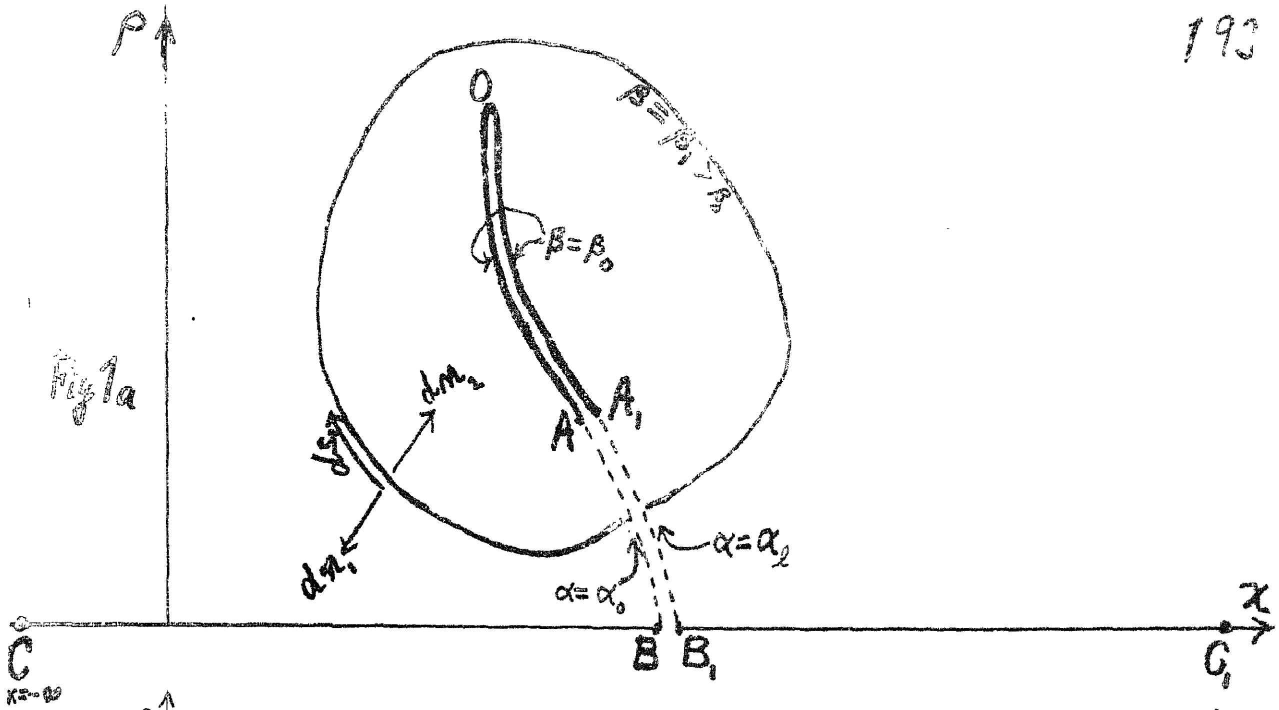


Fig 1a

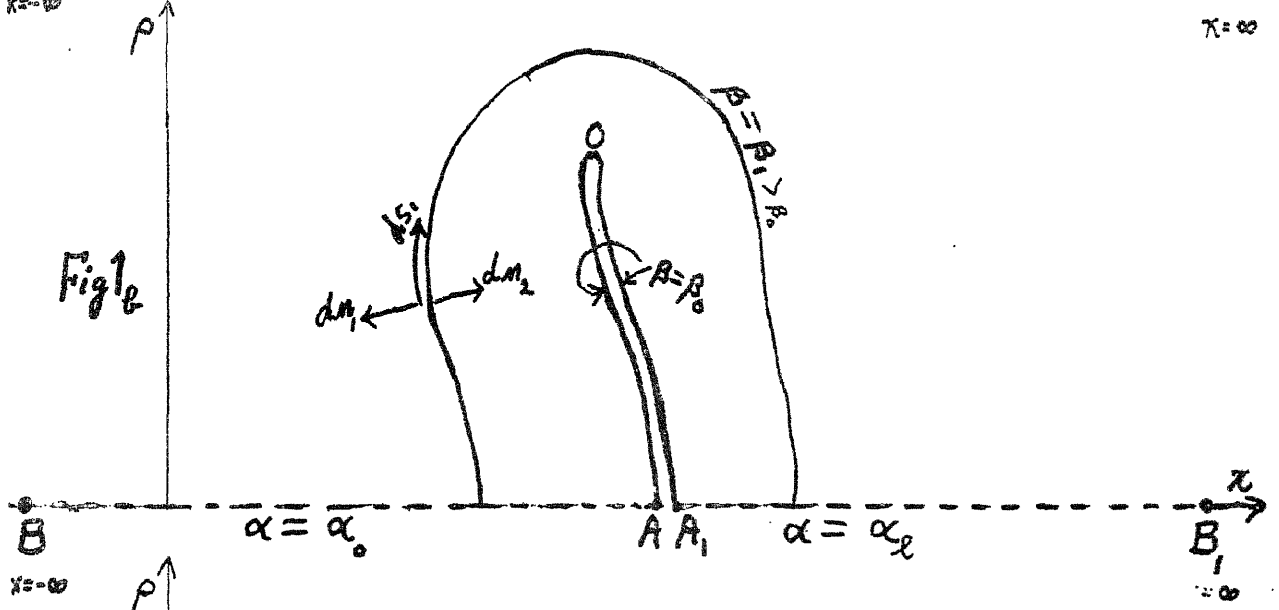


Fig 1b

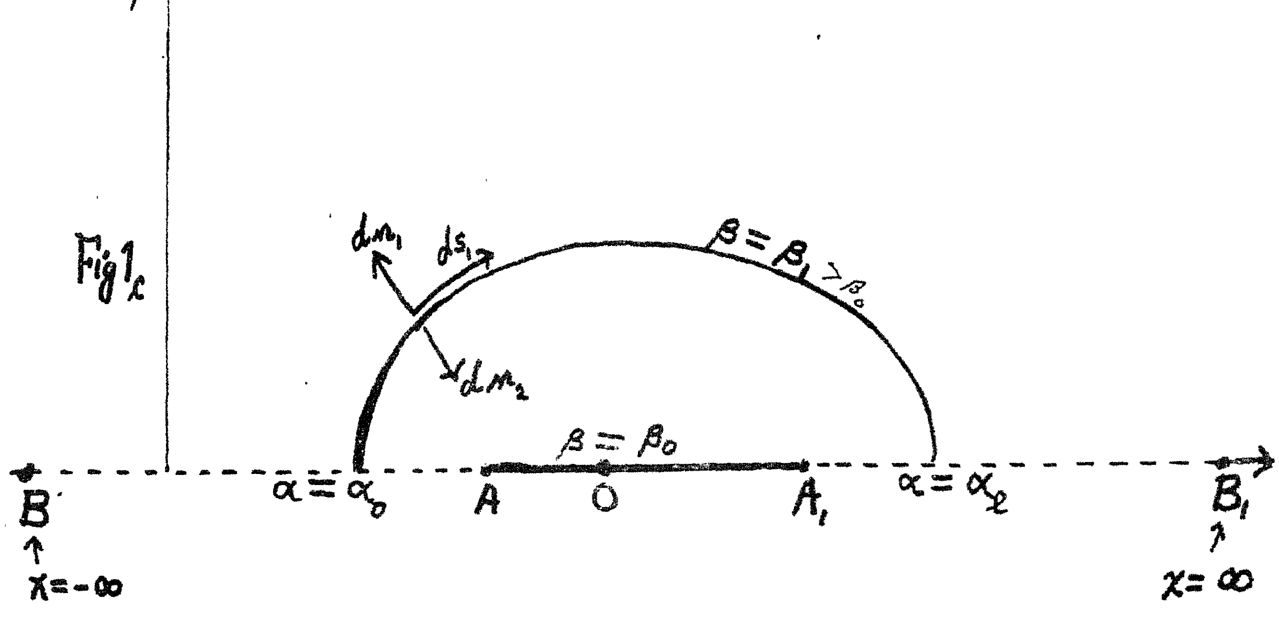


Fig 1c

as a dotted curve in each case. In fig 1e it is the x-axis, in fig 1c only part of it.

The following statements apply to all three figures

The entire locus $\beta=\beta_0$ is $\overline{AO}+\overline{OA_1}$ (heavy cut)

The entire locus $\alpha=\alpha_0$ is $\overline{AB}$ (dotted cut)

The entire locus $\alpha=\alpha_2$ is $\overline{A_1B_1}$ (dotted cut)

The curves $\beta=\beta_1$ begin perpendicularly on the side $\overline{AB}$ of the dotted cut where $\alpha=\alpha_0$ and end perpendicularly on the side $\overline{A_1B_1}$ where $\alpha=\alpha_2$. The relations (22) apply to each figure

The theory of integral equations indicates in general the existence of a complete set of normal functions $\phi_n^m(\alpha)$, $n=n_0, n_0+1, \cdots \infty$ for the range $\alpha_0<\alpha<\alpha_2$ which satisfy the homogeneous integral equation

$$25)\qquad \phi_n^m(\alpha)=\lambda_n^m\, 2\int_{\alpha_0}^{\alpha_2}\phi_n^m(\alpha_1)\, t(\alpha)\, t(\alpha_1)\, Q_{m-1/2}(g(\alpha,\alpha_1))\, d\alpha_1$$

where $t(\alpha)$ is some positive function of α on the curve $\beta=\beta_1$ and λ_n^m the characteristic associated with the function ϕ_n^m.

If the functions are normalized

$$26)\qquad \int^{\alpha_2}\phi_{n_1}^m(\alpha)\,\phi_{n_2}^m(\alpha)\,d\alpha=\delta_{n_1n_2}=1 \text{ if } n_1=n_2,\ =0 \text{ if } n_1\neq n_2$$

then the formal development theorem is

$$f(\alpha) = \sum_{n=n_0}^{\infty} \phi_n^m(\alpha) \int_{\alpha_0}^{\alpha_l} f(\alpha') \phi_n^m(\alpha') d\alpha' \quad \text{for} \quad \alpha_0 < \alpha < \alpha_l$$

Applying this to the function $2t(\alpha)\, t(\alpha_1)\, Q_{m-1/2}(g(\alpha,\alpha_1))$, considered as a function of α, while α_1 is a constant, gives by use of (25), the canonical expansion of the symmetrical nucleus

27) $$2\, t(\alpha)\, t(\alpha_1)\, Q_{m-1/2}(g(\alpha,\alpha_1)) = \sum_{n=n_0}^{\infty} \frac{\phi_n^m(\alpha)\, \phi_n^m(\alpha_1)}{\lambda_n^m}$$

The formal solution of the integral equation (24) is

28) $$\frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1)} = t(\alpha_1) \sum_{n=n_0}^{\infty} C_n\, \lambda_n^m\, \phi_n^m(\alpha_1) \quad \text{where} \quad C_n = \int_{\alpha_0}^{\alpha_l} t(\alpha_2)\, \bar{V}^m(\alpha_2)\, \phi_n^m(\alpha_2)\, d\alpha_2$$

Using this expansion for $\bar{\sigma}/h$ in eq (23) shows that the reduced potential of a simple distribution on the curve $\beta = \beta_1$ which has assigned values $\bar{V}^m(\alpha)$ there is given everywhere as a series of "normal potentials" $V_n^m(\alpha, \beta)$

29) $$V^m(\alpha, \beta) = \sum_{n=n_0}^{\infty} C_n\, V_n^m(\alpha, \beta)$$

where the normal potentials are defined everywhere

by

$$30)\quad \overset{m}{\underset{n}{U}}(\alpha,\beta) = 2\overset{m}{\underset{n}{\lambda}}\int_{\alpha_0}^{\alpha_\ell} t(\alpha_1)\,\overset{m}{\underset{n}{\phi}}(\alpha_1)\,Q_{m-1/2}(\,g(\alpha,\beta;\alpha_1,\beta_1)\,d\alpha_1$$

$$= 2\overset{m}{\underset{n}{\lambda}}\int_{0}^{\ell} t(\alpha_1)\,\overset{m}{\underset{n}{\phi}}(\alpha_1)\,h(\alpha_1)\,Q_{m-1/2}(\,g\,\alpha,\beta;\alpha_1,\beta_1))\,ds_1$$

This shows that $\overset{m}{\underset{n}{U}}(\alpha,\beta)$ is the (reduced) potential of a simple distribution on the curve $\beta=\beta_1$ whose density is

$$31)_a\quad \overset{m}{\underset{n}{\sigma}}(s) = \overset{m}{\underset{n}{\lambda}}\,t(\alpha)\,h(\alpha)\,\overset{m}{\underset{n}{\phi}}(\alpha)$$

The potential on the curve is by (30) and (25)

$$31)_b\quad \overset{m}{\underset{n}{U}}(\alpha,\beta_1\pm 0) = \frac{\overset{m}{\underset{n}{\phi}}(\alpha)}{t(\alpha)} \qquad \text{so that}$$

$$31)_c\quad \frac{\overset{m}{\underset{n}{\sigma}}(\alpha)}{h(\alpha)} = \overset{m}{\underset{n}{\lambda}}\,t(\alpha)\,\overset{m}{\underset{n}{\phi}}(\alpha) = t^2(\alpha)\,\overset{m}{\underset{n}{\lambda}}\,\overset{m}{\underset{n}{U}}(\alpha,\beta_1)$$

Eq. $(13)_e$ then becomes

$$32)\quad 4\pi\overset{m}{\underset{n}{\lambda}} = \int_{-\infty}^{\infty}dx\int_0^{\infty}d\rho\left\{\left(\frac{m\overset{m}{\underset{n}{U}}}{\rho}\right)^2 + \rho\left[\left(D_x\frac{\overset{m}{\underset{n}{U}}}{\sqrt{\rho}}\right)^2 + \left(D_\rho\frac{\overset{m}{\underset{n}{U}}}{\sqrt{\rho}}\right)^2\right]\right.$$

which shows that every $\overset{m}{\underset{n}{\lambda}}$ is positive.

Since the argument is retraceable from $(31)_c$ to (25) the problem of constructing the normal functions $\overset{m}{\underset{n}{\phi}}(\alpha)$ may be stated in a manner appropriate to the

methods of solving partial differential equations.

If a complete set of normal potentials or solutions $U_n^m(\alpha,\beta)$ $(n = n_0, n_0+1, \ldots \infty)$ of Euler's transformed equation (20) can be found, each of which satisfies the boundary conditions implied by the fact that it has no sources other than a simple distribution on the curve $\beta = \beta_1$, the density $\bar{\sigma}_n^m$ being defined by $4\pi\bar{\sigma}_n^m = -D_{n_1} U_n^m - D_{n_2} U_n^m$ and if density and potential at this curve satisfy a relation of type (31)$_c$ then the functions $\phi_n^m(\alpha)$ may be defined by (31)$_B$. Each of these will satisfy the homogenous integral equation (25) which is a special case of the integral relation (23) which is the potential $U_n^m(\alpha,\beta)$ of the density $\bar{\sigma}_n^m$. From the integral equation their orthogonality follows for (25) gives $(\lambda_{n_1}^m - \lambda_{n_2}^m)\int_{\alpha_0}^{\alpha_2} \phi_{n_1}^m(\alpha)\,\phi_{n_2}^m(\alpha)\,d\alpha = 0$

Since the normal potentials $U_n^m(\alpha,\beta)$ are constructed with reference to a particular curve $\beta = \beta_1$, each one will in general have different forms, $U_n^{o,m}(\alpha,\beta)$, when the point (α,β) is outside the curve $\beta = \beta_1$ and $U_n^{i,m}(\alpha,\beta)$ when it is inside. The normal functions $\phi_n^m(\alpha)$ and their eigen-values λ_n^m therefore belong to a given curve

and might be quite different from those belonging to any other curve $\beta = \beta_2$ of the same family.

The boundary conditions $(13)_a$ and $(13)_b$, which indicate no sources at infinity or on the x axis, must be satisfied by the external harmonics U_n^m.

The internal harmonics U_n^m must also satisfy $(13)_b$, the condition for no sources on that part of the x axis (if any) which is cut off by the curve $\beta = \beta_1$ (dotted in fig 1_b and 1_c). This condition is replaced in case of fig 1_a, by the requirement that the neighborhood of that part of the dotted cut which is enclosed by the curve $\beta = \beta_1$ must consist of regular points for these internal harmonics. The potential and its normal derivative must be continuous there, so it must be periodic in α with period $\alpha_2 - \alpha_0$. The remaining condition is that potential and normal derivatives must be continuous at the heavy cut $\overline{AOA_1}$ for all three figures.

The canonical expansion (27) of the nucleus may be considered as an addition-theorem for the function $Q_{m-1/2}(\varphi)$. To generalize it consider the case where the density $\sigma(\alpha)$ in (23) is a given function

of α. Developing $\frac{\bar\sigma(\alpha_1)}{t(\alpha_1)\,h(\alpha_1)}$ in the form $\sum_{n=n_0}^{\infty} a_n \Phi_n^m(\alpha_1)$ gives for the reduced potential (23) a series

$U^m(\alpha,\beta) = \sum_{n=n_0}^{\infty} \frac{a_n}{\lambda_n^m} U_n^m(\alpha,\beta)$ where $U_n^m(\alpha,\beta)$ is a normal potential defined by (30) which may be either U_n^{om} or U_n^{im} and

$a_n = \int_{\alpha_0}^{\alpha_2} \frac{\bar\sigma(\alpha')\,\Phi_n^m(\alpha')}{t(\alpha')\,h(\alpha')}\,d\alpha' = \int_0^{\ell} \frac{\bar\sigma(s')}{t(\alpha')}\,\Phi_n^m(s')\,ds'$.

This will represent a unit source at $s_1(\alpha_1,\beta_1)$ on the curve if we take the limit as $\Delta \to 0$ of the distribution defined by $\bar\sigma(s') = \frac{1}{\Delta}$ when $s_1-\Delta < s' < s_1+\Delta$
$= 0$ everywhere else,
which gives $a_n = \frac{\Phi_n^m(\alpha_1)}{t(\alpha_1)}$ in the limit $\Delta \to 0$.

The potential at any point (α,β) due to a unit source at (α_1,β_1) is $2\,Q_{m-1/2}(g(\alpha,\beta;\alpha_1\beta_1))$ so that

33) $2\,Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1)) = \frac{1}{t(\alpha_1)} \sum_{n=n_0}^{\infty} \frac{U_n^m(\alpha,\beta)\,\Phi_n^m(\alpha_1)}{\lambda_n^m}$ which is an

addition-theorem, which reduces to the canonical expansion (27) when the point α,β comes to a point (α,β)

on the curve from either side (as shown by $(31)_e$).
In this expansion $U_n^m(\alpha,\beta)$ is either $U_n^{om}(\alpha,\beta)$ or $U_n^{im}(\alpha,\beta)$.
While eq.(33) is valid for any position of the point (α,β) it is restricted to the case where the point (α_1,β_1) is on the particular curve $\beta=\beta_1$ with respect to which the normal functions $\phi_n^m(\alpha)$ and normal potentials $U_n^m(\alpha,\beta)$ have been (or are to be) constructed.

To extend the scope of the addition theorem let $W^m(\alpha,\beta)$ be a reduced potential of a simple distribution on $\beta=\beta_1$, which is equal to $2Q_{m-1/2}(g(\alpha,\beta;\alpha',\beta'))$ when (α,β) is any point inside the curve β_1 while (α',β') is a fixed point outside it so that $\beta\leq\beta_1$ and $\beta'>\beta_1$. Then by (33)

$$W^m(\alpha,\beta_1)=W^m(\alpha,\beta_1\pm 0)=2Q_{m-1/2}(g(\alpha,\beta_1;\alpha',\beta'))=$$

$$=\frac{1}{t(\alpha)}\sum_{n=m_0}^{\infty}\frac{U_n^m(\alpha',\beta')\,\phi_n^m(\alpha)}{\lambda_n^m}$$ by (33) since (33) is symmetric and holds if at least one of the points is on the curve.

Hence by (29) W^m is given at every point (α,β) by

$$W^m(\alpha,\beta)=\sum_{n=m_0}^{\infty}C_n U_n^m(\alpha,\beta)$$ where

$$C_n=\int_{\alpha_0}^{\alpha_e}t(\alpha_1)\phi_n^m(\alpha_1)W^m(\alpha_1,\beta_1)d\alpha_1=\int_{\alpha_0}^{\alpha_e}\phi_n^m(\alpha_1)d\alpha_1\sum_{K=m_0}^{\infty}\frac{U_K^m(\alpha',\beta')\,\phi_K^m(\alpha_1)}{\lambda_K^m}$$

$$=\sum_{K=m_0}^{\infty}\frac{U_K^m(\alpha',\beta')}{\lambda_K^m}\int_{\alpha_0}^{\alpha_e}\phi_n^m(\alpha_1)\phi_K^m(\alpha_1)d\alpha_1=\frac{U_n^m(\alpha',\beta')}{\lambda_n^m}.$$

This gives

$$34) \quad 2Q_{m-1/2}(g(\alpha,\beta;\alpha';\beta') = \sum_{n=n_0}^{\infty} \frac{U_n^m(\alpha,\beta)\, U_n^m(\alpha';\beta')}{\lambda_n^m}$$ provided that

(α,β) and (α',β') are on opposite sides of the curve $\beta=\beta_1$, or if one or both are on it.

This is the most general form of the addition-theorem.

It may be noted that in exceptional cases the above formulation breaks down. For example if $\alpha_2-\alpha_0\to\infty$ the eigen-values λ_n^m may merge into a continuous spectrum and the expansion of a function in a series of normal functions, is replaced by an integral representation.

This formal discussion may be ended with the suggestion that it would be interesting to find out whether or not each of the normal functions $\phi_n^m(\alpha)$ associated with a given curve, $\beta=\beta_1$, could be a solution of one ordinary differential equation, say

$$35) \quad \frac{d}{d\alpha}\left[R(\alpha,\beta_1)\frac{d}{d\alpha}\left(\frac{\phi^m(\alpha)}{t(\alpha)}\right)\right] + \left[\left(\tfrac{1}{4}-m^2\right)P(\alpha,\beta_1) + \nu_n^m Q(\alpha,\beta_1)\right]\phi^m(\alpha) = 0$$

where n appears only in the constant ν_n^m. In this general

form, containing the constant β_1, the normal functions $\phi_n^m(\alpha)$ could be quite different functions of α, if defined with respect to another curve $\beta = \beta_2$ of the same family. The answer to the above question would open (or close) the way for a reduction of Euler's equation to ordinary differential equations in a manner different from that next to be considered, which is by use of the so-called separable coordinate pairs.

(b) Integral equation with separable coordinates.

With "separable" coordinate pairs (α, β) it is found that the normal functions for $\beta = \beta_1$ are the same as for any other member of the family β = constant. Only the characteristics λ_n^m vary.

Such coordinates may be defined as those in which Euler's transformed equation (20) has solutions of the form $V^m(\alpha,\beta) = \mathcal{U}^m(\alpha) \cdot \mathcal{V}^m(\beta)$. On substituting this in (20) it becomes $(\frac{1}{4} - m^2)\, S(\alpha,\beta) = \frac{\mathcal{U}''^m(\alpha)}{\mathcal{U}^m(\alpha)} + \frac{\mathcal{V}''^m(\beta)}{\mathcal{V}^m(\beta)}$ so the necessary and sufficient condition for separability is that the transformation (17), $x + i\rho = f(\alpha + i\beta)$, be such as to

make $1/h^2\rho^2$ have the form

36)$_a$ $S(\alpha,\beta) \equiv \dfrac{1}{h^2\rho^2} = p(\alpha) + q(\beta)$, which may also be written

36)$_b$ $D_\alpha D_\beta S = 0$, since the general integral of this is (36)$_a$. In that case Euler's eq (20) is reduced to the two ordinary differential equations

37)$_a$ $u''^m(\alpha) + [(1/4 - m^2)\,p(\alpha) + \nu^m]\,u^m(\alpha) = 0.$

37)$_b$ $v''^m(\beta) + [(\frac{1}{4} - m^2)\,q(\beta) - \nu^m]\,v^m(\beta) = 0.$

If $v^{o\,m}(\beta)$ is a solution of (37)$_b$ which has no singularities for $\beta_1 \leq \beta$ and $v^{i\,m}(\beta)$ one which has none for $\beta_0 \leq \beta \leq \beta_1$ then the external and internal harmonics with respect to the curve $\beta = \beta_1$ have the form

38)$_a$ $U^{o,m}(\alpha,\beta) = u^m(\alpha).\,v^{o,m}(\beta) \qquad \beta_1 \leq \beta$

38)$_b$ $U^{i,m}(\alpha,\beta) = u^m(\alpha).\,v^{i\,m}(\beta) \qquad \beta_0 \leq \beta \leq \beta_1$

The Bernoulli parameter ν^m in eq (37)$_a$ must be so chosen that its solutions $u^m(\alpha)$ will make the potentials (38) that of a simple distribution on the curve $\beta = \beta_1$.

In the case represented by fig 1a this is Hill's type of boundary value problem, requiring solutions of (37)$_a$ which are periodic with period $\alpha_2 - \alpha_0$, since the dotted cut AB, A_1B_1 must consist of ordinary points for the

potentials $(38)_a$ as well as $(38)_b$. This requires that $u(\alpha_0) = u(\alpha_\ell)$ and $u'(\alpha_0) = u'(\alpha_\ell)$ so that complete periodicity results, since $(37)_a$ is of second order.

In fig 1_b the dotted cut has become the entire x axis and in fig 1_c part of it. The boundary value problem in these cases (which determine the eigen-values ν_n^m and their eigen-functions $u_n^m(\alpha)$) consists in satisfying the condition $(13)_b$ for no sources on the x axis. The duty of insuring no sources at infinity devolves in these cases upon the function $v^{o\,m}(\beta)$ which is regular for the value of β which represents spatial infinity. Similarly $v^{i\,m}(\beta)$ takes care of the heavy cut $\overline{AOA_1}$.

The set of functions thus determined as solutions of $(37)_a$ say $u_n^m(\alpha)$, $n = n_0, n_0+1, \ldots \infty$ will be an orthogonal set for the range $\alpha_0 < \alpha < \alpha_\ell$, for if ν_n^m are the characteristics we find from $(37)_a$

$$39) \quad \left(\nu_{n_1}^{m\,2} - \nu_{n_2}^{m\,2}\right) \int_{\alpha_0}^{\alpha_\ell} u_{n_1}^m(\alpha)\, u_{n_2}^m(\alpha)\, d\alpha = \left[u_{n_1}^m(\alpha_0)\, u_{n_2}^{m\,\prime}(\alpha_0) - u_{n_1}^{m\,\prime}(\alpha_0)\, u_{n_2}^m(\alpha_0)\right]$$

$$- \left[u_{n_1}^m(\alpha_\ell)\, u_{n_2}^{m\,\prime}(\alpha_\ell) - u_{n_1}^{m\,\prime}(\alpha_\ell)\, u_{n_2}^m(\alpha_\ell)\right]$$

which is zero if $n_1 \neq n_2$. In cases like fig 1_a it vanishes because

of the periodicity of the functions, and in cases like fig 1_b ad 1_c because of the boundary condition $(13)_b$.

In the following we assume the set is normalized, so that

40) $$\int_{\alpha_0}^{\alpha_2} \mathcal{U}_{n_1}^{m}(\alpha)\, \mathcal{U}_{n_2}^{m}(\alpha)\, d\alpha = \delta_{n_1, n_2}.$$

It is evident that the formula of the preceding part (a) are here applicable with $t(\alpha) \equiv 1$. The external and internal form of the normal solutions with respect to the "closed" curve $\beta = \beta_1$ may be written

$41)_a$ $$U_n^{o,m}(\alpha, \beta) = \mathcal{U}_n^{m}(\alpha) \frac{V_n^{om}(\beta)}{V_n^{om}(\beta_1)} \quad \text{outside} \quad \text{where} \quad \beta_1 \leq \beta$$

$41)_b$ $$U_n^{im}(\alpha, \beta) = \mathcal{U}_n^{m}(\alpha) \frac{V_n^{im}(\beta)}{V_n^{im}(\beta_1)} \quad \text{inside} \quad \text{where} \quad \beta_0 \leq \beta \leq \beta_1$$

for $n = n_0, n_0+1, \dots \infty$.

These normal solutions may be called the (external) and (internal) harmonic continuations of the normal functions $\mathcal{U}_n^{m}(\alpha)$ to which they reduce on the curve β_1. Their only source is a simple distribution on this curve. With the particular convention adopted in figure $1_{a,b,c}$ for the direction of increasing β, $dn_1 = \frac{d\beta}{h(\alpha)} = \frac{d\beta}{h(\alpha,\beta_1)} = -dn_2$ this density is given by

$$\frac{4\pi \bar{\sigma}_n^{m}(\alpha)}{h(\alpha)} = \mathcal{U}_n^{m}(\alpha) \frac{\left[V_n^{om} V_n^{\prime im} - V_n^{\prime om} V_n^{im} \right]_{(\beta_1)}}{V_n^{om}(\beta_1)\; V_n^{im}(\beta_1)}$$

Since $\mathcal{V}^i$ and $\mathcal{V}^o$ are both solutions of $(37)_b$ the numerator is a constant (i.e. independent of β_1). Hence we may write

$$42)_a \quad \frac{4\pi\bar{\sigma}^m_n(\alpha)}{h(\alpha,\beta_1)} = 4\pi\lambda^m_n(\beta_1)\,\mathcal{U}^m_n(\alpha) = \frac{\gamma^m_n}{\mathcal{V}^{om}_n(\beta_1)\,\mathcal{V}^{im}_n(\beta_1)}\cdot\mathcal{U}^m_n(\alpha)$$

where

$$42)_b \quad \lambda^m_n(\beta_1) \equiv \frac{\gamma^m_n}{4\pi\,\mathcal{V}^{om}_n(\beta_1)\,\mathcal{V}^{im}_n(\beta_1)}$$

where the constant γ^m_n is defined for the case where β increases outward from β_0 as in figures 1a, b, c, by

$$42)_c \quad \gamma^m_n \equiv \left[\mathcal{V}^{om}_n\mathcal{V}'^{im}_n - \mathcal{V}'^{om}_n\mathcal{V}^{im}_n\right]_{\beta_1} = \text{independent of } \beta_1$$

In case β decreases outward from β_0 this would be replaced by

$$42)_d \quad \gamma^m_n \equiv \left[\mathcal{V}^{im}_n\mathcal{V}'^{om}_n - \mathcal{V}'^{im}_n\mathcal{V}^{om}_n\right]$$

Both forms of (41) are included in the integral

$$U^m_n(\alpha,\beta) = 2\int_{\alpha_0}^{\alpha_e}\frac{\bar{\sigma}^m_n(\alpha_1)}{h(\alpha_1,\beta_1)}\,Q_{m-\frac{1}{2}}\big(q(\alpha,\beta;\alpha_1,\beta_1)\big)\,d\alpha_1$$

Hence introducing the above values of U^m_n and of $\bar{\sigma}^m_n/h$ in this integral gives a homogeneous integral equation satisfied by $\mathcal{U}^m_n$.

$$43)\quad \int_{\alpha_0}^{\alpha_2} \mathcal{U}_n^m(\alpha_1)\, Q_{m-\frac{1}{2}}(g(\alpha,\beta;\alpha_1,\beta_1))\, d\alpha_1 = \frac{2\pi\, \mathcal{V}_n^{om}(\beta_1)\, \mathcal{V}_n^{im}(\beta)}{\gamma_n^m}\, \mathcal{U}_n^m(\alpha) \quad \text{if} \quad \beta_1 \leq \beta$$

$$= \frac{2\pi\, \mathcal{V}_n^{om}(\beta_1)\, \mathcal{V}_n^{im}(\beta)}{\gamma_n^m}\, \mathcal{U}_n^m(\alpha) \quad \text{inside} \quad \beta_0 \leq \beta \leq \beta_1$$

Considering $Q_{m-\frac{1}{2}}(q)$ as a function of α_1 where α, β and β_1 are fixed, it may be developed in a series of normal functions $\mathcal{U}_n^m(\alpha)$ and its Fourier coefficients are given by this integral equation. This gives

$$44)\quad Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1)) = 2\pi \sum_{n=n_0}^{\infty} \frac{\mathcal{U}_n^m(\alpha)\, \mathcal{V}_n^{om}(\beta)\cdot \mathcal{U}_n^m(\alpha_1)\, \mathcal{V}_n^{im}(\beta_1)}{\gamma_n^m}$$

when $(\alpha\,\beta)$ is outside the curve $\beta = \beta_1$. Since g is symmetric in the two points, β and β_1 are to be interchanged in this when α, β is inside the curve.

The case of an open curve is a limiting case of this so that if we place $\beta_1 = \beta_0$ in (44) the expansion is then valid when α, β is any point, corresponding to any point in the x, ρ half plane. The three cases of fig 1 then show where the singularities of any external harmonic are located (i.e. on the line AOA, in every case).

Examples are given in section X in many of which this addition-formula takes the form of an integral.

3. The spatial interpretations of reduced potential.

If the change of independent variables known as "inversion" is applied to Laplaces three-dimensional equation, the original form is regained by also making a transformation of the dependent variable or potential. For Euler's equation the last step has already been made. To keep to real independent variables we consider only inversions with respect to circles which are centered on the real axis, and this may be represented by a real homographic transformation between two complex variables.

If x, ρ and $x'\rho'$ are two half-planes (ρ and ρ' both positive) and $z = x + i\rho$, $z' = x' + i\rho'$, the transformation

45)$_a$ $$x = x_0 - \frac{c^2(x'-x_0')}{(x'-x_0')^2+\rho'^2} \quad \text{and} \quad \rho = \frac{c^2\rho'}{x'-x_0')^2+\rho'^2} \qquad \text{where } c^2 > 0$$

may also be written

45)$_b$ $$(z - x_0)(z' - x_0') = -c^2$$

or

45)$_c$ $$z = \frac{Az'+B}{Cz'+D} \quad \text{where } AD - BC \neq 0, \text{ the constants being real.}$$

The effect of this transformation on Euler's eq.(5)

is merely to replace the real independent variables x, ρ by the real pair x', ρ', assuming that x_0, x_0' and c are real in $(45)_c$. The reduced potential U^* remains the dependent variable. Also the function g is invariant

$g \equiv 1 + \frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1} = 1 + \frac{(x'-x_1')^2+(\rho'-\rho_1')^2}{\rho'\rho_1'}$ where (x', ρ') is the transform of (x, ρ) and (x_1', ρ_1') that of (x_1, ρ_1)

If the $x\rho$, half-plane is mapped upon some area in the u-plane by the transformation

$z = f(u)$ and the z half-plane is inverted into the z' half-plane by the transformation $(45)_c$, then the z' half-plane is represented upon the same area of the u-plane by the transformation

$z' = F(u)$ where $f(u)$ and $F(u)$ are related by

46) $$f(u) = \frac{AF(u)+B}{CF(u)+D}$$

Consequently the function $S(\alpha, \beta) = \frac{1}{\rho^2 h^2}$ in Euler's transformed eq (20) is invariant to the substitution (46). Any solution U of (20) is therefore capable of an interpretation as a (reduced) potential either on the $x\rho$ or $x'\rho'$ half plane. This is true whether the coordinates are of the "separable" class or not. (In returning to

the ordinary potentials the factor $\frac{\cos m(\phi-\phi_m)}{\sqrt{\rho}}$ is affixed to reduced potential U^m).

The applications in section X are arranged to illustrate the various spatial interpretations of the same potential expressed in invariant form (i.e. in terms of α and β).

It may be noted that the formal aspect of the transformation $(45)_a$ is larger than the view presented here for beginning with the real variables x, ρ we could admit complex values for the constants x_0, x_0' and c^2. The new variables x', ρ' would then be complex but Euler's equation would still be invariant. In that case, $(45)_b$ and $(45)_c$ are derivable from $(45)_a$ but not conversely, if $z' \equiv x'+i\rho'$ and $z \equiv x+i\rho$.

4. The class of separable coordinates for the potential equation.

If the variables x, ρ in eq (3) be replaced by two others $\lambda_1(x,\rho)$, $\lambda_2(x,\rho)$, where λ_1 and λ_2 are orthogonal but not necessarily conjugate functions, we may seek for solutions of the form $V^m = T(\lambda_1, \lambda_2)\, U(\lambda_1)\, V(\lambda_2)$ where T is a weighting factor to be found while u and v satisfy homogeneous linear differential equations of second order. The result is that such solutions are only possible when $\lambda_1 = \lambda_1(\alpha)$ so λ_2 must be $\lambda_2(\beta)$ where α and β are conjugate functions so that nothing is gained by using λ_1, λ_2 and we take (α, β) as coordinates. The transformation from (x,ρ) to (α,β) is conformal. It is then found that $T = \rho^{-\frac{1}{2}}$ and the necessary and sufficient condition for the existence of such solutions is the equation (36) The conclusion is that whenever the potential equation is separable, the same is true of Euler's eq (5) Hence the interest attached to the study of those conjugate functions which satisfy the condition (36). In the case of certain elementary pairs, corresponding to

circular cylindrical, spherical, spheroidal and parabolic coordinates, the potential equation (3) is directly separable without the necessity of passing to Euler's equation by the change of dependent variable $V^{**} = \rho^{-\frac{1}{2}} U^{**}$. These present no exception since in all of them ρ is of the form $f_1(\alpha) \cdot f_2(\beta)$ so they are also separable variables for Euler's equation.

The equation of transformation

47)$_a$ $\quad z \equiv x + i\rho) = f(u) \equiv f(\alpha + i\beta)$ implies that

$\bar{z} \equiv x - i\rho = f_1(\bar{u}) \equiv f_1(\alpha - i\beta)$ where $f_1(\alpha - i\beta)$ is the conjugate of $f(\alpha + i\beta)$ which is the same as $f(\alpha - i\beta)$ only in the case of so-called "real functions" of the complex variable $\alpha + i\beta$ such as may be defined by powers of $\alpha + i\beta$ with real coefficients.

Hence we may write

47)$_b$
$$\begin{cases} \dfrac{1}{h_\beta^2} \equiv \left|\dfrac{dz}{du}\right|^2 = |f'(u)|^2 = f'(u)\, f_1'(\bar{u}) \qquad \text{or} \\[2ex] f'(u)^2 = \dfrac{1}{h_\beta^2} e^{2i\Theta} = (D_\beta \rho)^2 - (D_\alpha \rho)^2 + i\, 2\, D_\alpha \rho \cdot D_\beta \rho \end{cases}$$

Hence, considered as functions of x, ρ the pair $\log \frac{1}{h_\beta}$ and Θ are conjugate functions, as also are the pair $(D_\beta \rho)^2 - (D_\alpha \rho)^2$, $\quad 2 D_\alpha \rho \cdot D_\beta \rho$. Consequently there are

the two equations

$47)_c$ $(\mathcal{D}_x^2 + \mathcal{D}_\rho^2)\log h_g \equiv h_g^2(\mathcal{D}_\alpha^2 + \mathcal{D}_\beta^2)\log h_g = 0$

and

$47)_d$ $(\mathcal{D}_x^2 + \mathcal{D}_\rho^2)(\mathcal{D}_\alpha\rho \cdot \mathcal{D}_\beta\rho) = 0$

Also

$47)_e$ $\mathcal{D}_\alpha^2 x = \mathcal{D}_\rho(\mathcal{D}_\alpha\rho \cdot \mathcal{D}_\beta\rho)$

and

$47)_f$ $\mathcal{D}_\alpha^2 \rho = \mathcal{D}_x(\mathcal{D}_\alpha\rho \cdot \mathcal{D}_\beta\rho)$

If $T(x)$ is any function of x only we derive the formula

$47)_g$ $\mathcal{D}_\alpha\mathcal{D}_\beta\left(\frac{T(x)}{h_g^2}\right) = \frac{2}{h_g^2}\mathcal{D}_x\left[T^{\frac{1}{2}}(x)\,\mathcal{D}_x\left(T^{\frac{1}{2}}(x)\,\mathcal{D}_\alpha x \cdot \mathcal{D}_\beta x\right)\right]$

and similarly since $\mathcal{D}_\alpha x \cdot \mathcal{D}_\beta x = -\mathcal{D}_\alpha\rho \cdot \mathcal{D}_\beta\rho$

$47)_h$ $\mathcal{D}_\alpha\mathcal{D}_\beta\left(\frac{T(\rho)}{h_g^2}\right) = \frac{2}{h_g^2}\mathcal{D}_\rho\left[T^{\frac{1}{2}}(\rho)\,\mathcal{D}_\rho\left(T^{\frac{1}{2}}(\rho)\,\mathcal{D}_\alpha\rho \cdot \mathcal{D}_\beta\rho\right)\right]$

The term $S(\alpha,\beta)$ in Euler's eq (20) is defined by

$47)_i$ $S(\alpha,\beta) \equiv \frac{1}{\rho^2 h_g^2} = \frac{(\mathcal{D}_\alpha\rho)^2 + (\mathcal{D}_\beta\rho)^2}{\rho^2} = -(\mathcal{D}_\alpha^2 + \mathcal{D}_\beta^2)\log\rho$

Hence taking $T(\rho) = 1/\rho^2$ in $(47)_h$ we obtain the

formula

$$47)_j \quad \mathcal{D}_\alpha \mathcal{D}_\beta S(\alpha,\beta) = \frac{2}{h_f^2} \mathcal{D}_\rho \left[\frac{1}{\rho} \mathcal{D}_\rho \left(\frac{\mathcal{D}_\alpha \rho \cdot \mathcal{D}_\beta \rho}{\rho} \right) \right]$$

The function $S(\alpha,\beta)$ arose from a particular transformation $z = f(u)$ which carries Eulers equation (5) into the form (20), but we have seen in (46) that this is only one of a triple infinity of transformations $z' = F(u)$ where $f(u) = (A\,F(u) + B)/(C\,F(u) + D)$ with real constants which would have the same effect. Since $2i\rho = f(u) - f_1(\bar{u})$ the transformation $z' \equiv x' + i\rho' = F(u)$ gives $2i\rho' = F(u) - F_1(\bar{u})$ so the definition $(47)_a$ could also be written

$$47)_K \quad S(\alpha,\beta) \equiv \frac{1}{\rho^2 h_f^2} = \frac{-4\, f'(u)\, f_1'(\bar{u})}{[f(u) - f_1(\bar{u})]^2} = \frac{-4\, F'(u)\, F_1'(\bar{u})}{[F(u) - F_1(\bar{u})]^2} = \frac{1}{\rho'^2 h_F^2}$$

which is derivable from (46) when and only when its constants A, B, C, D are real. The group of transformations (46) belongs to S which could be defined in terms of any one of them. There should be some way of characterizing S (in addition to its being a positive real), which is independent of any particular member of the group, which would be necessary and sufficient to insure that an equation of form (20) may be

be the transform of eq(5). This relation is found by taking the log of $(47)_i$, applying $(D_\alpha^2 + D_\beta^2)$ and using $(47)_e$ and $(47)_i$. This eliminates x and ρ and gives in invariant form the non-linear partial differential equation

48) $(D_\alpha^2 + D_\beta^2)\log S(\alpha,\beta) = 2S(\alpha,\beta)$

which every positive real S must satisfy in order that (20) may be the transform of Euler's eq(5). It may also be shown to be sufficient, that is every positive real solution S determines a group of transformations by which (5) and (20) are carried one into the other.

We may now consider separable coordinates and instead of the equation $(36)_a$ we may write the more symmetrical form

49) $S(\alpha,\beta) = 4[p(2\alpha) - q(2i\beta)]$ where p and q are real functions of their argument which depend upon the transformation group. Eulers eq (20) then has solutions of the form

50) $\overset{m}{V}(\alpha,\beta) = \overset{m}{u}(2\alpha)\,\overset{m}{v}(2i\beta)$ where

$51)_a$ $\overset{m}{u}{}''(2\alpha) + \left[(\tfrac{1}{4}-m^2)\,p(2\alpha) + \nu\right]\overset{m}{u}(2\alpha) = 0$

and

$51)_b$ $\overset{m}{v}{}''(2i\beta) + \left[(\tfrac{1}{4}-m^2)\,q(2i\beta) + \nu\right]\overset{m}{v}(2i\beta) = 0$

These are the equivalent of the equations for the dependent variables u^m, v^m of the pair of ordinary equations $37)_a$ $(37)_b$ (The real dependant variable $\overset{m}{u}(2\alpha)$ in (50) and $(51)_a$ or in $(37)_a$ will not be confused with the complex variable $u \equiv \alpha + i\beta$)

In following out the consequences of the assumption (49) we may first find whether it is compatible with the fundamental equation (48) which every S must satisfy. It is found to be compatible with (48) and the next step will be to find a form of differential equation which must be satisfied by every transformation function $f(u)$ of the group belonging to such an S.

Substitution of (49) in (48) leads to an integrable system of equations which amount to requiring $p(2\alpha)$ and $q(2i\beta)$ to satisfy the equations

$52)_a$ $$p'^2 = 4(p^3 + 3b_1 p^2 + 3b_2 p + b_3)$$

$52)_b$ $$q'^2 = 4(q^3 + 3b_1 q^2 + 3b_2 q + b_3)$$

where b_1, b_2 and b_3 are arbitrary real constants

These may be written

$53)_a$ $$p'^2 = 4(p+b_1)^3 - g_2(p+b_1) - g_3$$

$53)_b$ $$q'^2 = 4(q+b_1)^3 - g_2(q+b_1) - g_3$$

where $g_2 = 12(b_1^2 - b_2)$, $g_3 = 4(3b_1b_2 - 2b_1^3 - b_3)$

The choice of b_1 is a matter of expediency depending

upon the form in which we desire to integrate the equations. Since p and q occur in (49) only in their difference we could write that equation

$$S = 4[\,p + e_1 - (q + e_1)\,],$$ hence there is no loss of generality in taking

$$\left.\begin{aligned} &54)_a \quad p'^2 = 4p^3 - g_2 p - g_3 \\ &54)_b \quad q'^2 = 4q^3 - g_2 q - g_3 \end{aligned}\right\} \text{ where } g_2 \text{ and } g_3 \text{ are any real constants}$$

so that the integrals are

$$55)_a \qquad p(2\alpha) = \wp(2(\alpha - \alpha_0))$$

$$55)_b \qquad q(2i\beta) = \wp(2i(\beta - \beta_0))$$

where $\wp(u)$ denotes Weierstrass's $\wp$-function, $\wp(u, g_2\, g_3)$, formed with the real invariants g_2 and g_3.

In the equations of meridian curves given below, the functions $p(2\alpha)$ and $q(2i\beta)$ satisfy the Weierstrassian normal form of equation $(54)_a$ $(54)_b$. In applications it is often more convenient to use Jacobian elliptic functions ~~in~~ which the equation of transformation gives S by means of simpler addition theorems than that belonging to $\wp(u)$. In that case no particular advantage is attached to the system (54) over (52).

The next step is to derive a differential equation which must be satisfied by the

transformation function $f(x)$. Eliminating the undetermined functions $p(2\alpha)$ and $q(2i\beta)$ by applying $D_\alpha D_\beta$ to eq(49) gives the reducibility condition (36)$_e$

56) $D_\alpha D_\beta S(\alpha,\beta) = 0$ or by (47)$_j$

57)$_a$ $D_\rho \left[\frac{1}{\rho} D_\rho \left(\frac{D_\alpha \rho \cdot D_\beta \rho}{\rho} \right) \right] = 0$

Integrating this partially, the "constants" of integration will be functions of x, which however are of very limited generality, since (47)$_d$ must be satisfied. This gives

57)$_b$ $D_\alpha \rho \cdot D_\beta \rho = 2(a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3)\rho - 2(a_0 x + a_1)\rho^3$

where a_0, a_1, a_2 and a_3 are arbitrary real constants. This equation is necessary and sufficient that (56), and hence (49), be true for the steps are retraceable. Appling D_ρ to (57)$_b$ gives $D_\alpha^2 x$ by (47)$_c$. Similarly applying D_x to (57)$_b$ gives $D_\alpha^2 \rho$ by (47)$_f$. Hence

58)$_a$ $D_\alpha^2 x = 2(a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3) - 6(a_0 x + a_1)\rho^2$

58)$_b$ $D_\alpha^2 \rho = 6(a_0 x^2 + 2a_1 x + a_2)\rho - 2a_0 \rho^3$

which would also have been obtained had (57)$_b$ contained an additive constant. Hence we cannot recover (57)$_b$ starting from (58)$_a$ and (58)$_b$ without

placing the condition that $D_\alpha\rho\cdot D_\beta\rho$ vanishes when $\rho=0$, that is by 47)$_8$

59) $\quad f'^{2}(u)$ is real when $\rho=0$

Since $z\equiv x+i\rho = f(u)$, $\frac{d^2z}{du^2} = f''(u) = D_\alpha^2 x + i D_\alpha^2\rho$, so that the two real equations (58) are equivalent to the single equation

60) $$\frac{d^2z}{du^2} = f''(u) = 2(a_0 z^3 + 3a_1 z^2 + 3a_2 z + a_3)$$

Multiplying this by $2f'(u)du = 2\left(\frac{dz}{du}\right)du = 2dz$ and integrating, gives

61) $$\left(\frac{dz}{du}\right)^2 = f'^{2}(u) = R(z) \equiv a_0 z^4 + 4a_1 z^3 + 6a_2 z^2 + 4a_3 z + a_4$$
$$= a_0 f^4 + 4a_1 f^3 + 6a_2 f^2 + 4a_3 f + a_4$$

where the new constant a_4 also must be real by (59). This is satisfied when $f(u)$ is any elliptic function of u of second order.

The conclusion is that (α,β) will be a separable coordinate pair when $z = x+i\rho = f(u) = f(\alpha+i\beta)$ whenever $f(u)$ is such a function of u that $f'^{2}(u) = R(f) =$ a real quartic in f. The transformation (-6) gives the same character to F, i.e. $F'^{2}(\bar{u}) = T(F)$ where T is a real quartic in F having the same real invariants g_2 and g_3 as R. These are

62) $$\begin{cases} g_2 = a_0a_4 - 4a_1a_3 + 3a_2^2 \\ g_3 = a_2(a_0a_4 + 2a_1a_3) - a_0a_3^2 - a_1^2a_4 - a_2^3 \end{cases}$$

The elementary transformations which lead to separable coordinates, correspond to cases in which the four roots of $R(z)$ are not all distinct; the discriminant, $g_2^3-27g_3^2$, then being zero.

If c is any root, real or complex, of $R(z)=0$, the substitution

63)$_a$ $$z = c + \frac{R'(c)}{4\left[y - \frac{R''(c)}{24}\right]}$$

gives

63)$_b$ $$R(z) = \frac{R'^2(c)\left[4y^3-g_2y-g_3\right]}{16\left[y-\frac{R''(c)}{24}\right]^4}$$

The root $z=c$ of the quartic, corresponds to $y=\infty$ in (63)$_a$; the other three roots of the quartic correspond to $y=e_1, e_2, e_3$, these being the roots of the reducing cubic

63)$_c$ $4e^3-g_2e-g_3=0$.

The equation (61) transforms into

63)$_d$ $\left(\frac{dy}{du}\right)^2 = 4y^3-g_2y-g_3=0$ whose solution is

$y=\wp(u+u_0)$ so the general solution of (61) may be taken in the form

64) $$z=f(u)=c+\frac{R'(c)}{4\left[\wp(u+u_0)-\frac{R''(c)}{24}\right]}$$

where the constant of integration u_0 may be taken

as zero to obtain a particular type of transformation.

One of the simplest groups of "separable" transformations is found when the quartic has some real roots. Taking c as one of these, then c, $R'(c)$ and $R''(c)$ are real, so that the solution (64) gives $z = f(u)$ = a real homographic function of $\wp(u)$. Hence this group is

65)$_f$ $\quad z' = F(u) = \wp(u)$

65)$_a$ $\quad z = f(u) = \dfrac{A\,\wp(u)+B}{C\,\wp(u)+D} = \dfrac{Az'+B}{Cz'+D}$ (real constants A, B, C, D)

Since $z' = F(u) = \wp(u)$, $F'(u) = \wp'(u)$

$\bar{z}' = F_1(\bar{u}) = \wp(\bar{u})$, $F_1'(\bar{u}) = \wp'(\bar{u})$.

Hence by 47)$_K$ and (49)

$$S(\alpha,\beta) = 4\left[p(2\alpha) - q(2i\beta)\right] = -\frac{4\,\wp'(\alpha+i\beta)\,\wp'(\alpha-i\beta)}{\left[\wp(\alpha+i\beta) - \wp(\alpha+i\beta)\right]^2}$$

By use of the general formula

66) $$\wp(2u) - \wp(2v) = \frac{-\wp'(u+v)\,\wp'(u-v)}{\left[\wp(u+v) - \wp(u-v)\right]^2}$$

it is found that $p(2\alpha) = \wp(2\alpha)$ and $q(2i\beta) = \wp(2i\beta)$ so the ordinary equations (51) become for the group of transformations (65)

67)$_a$ $$\mathcal{U}''^{m}_{(2\alpha)} + \left[\left(\tfrac{1}{4} - m^2\right)\wp(2\alpha) + \nu\right]\mathcal{U}^{m}_{(2\alpha)} = 0$$

67)$_b$ $$\mathcal{V}''^{m}_{(2i\beta)} + \left[\left(\tfrac{1}{4} - m^2\right)\wp(2i\beta) + \nu\right]\mathcal{V}^{m}_{(2i\beta)} = 0$$

The equation of the meridian curves α = constant or β = constant, in the x', ρ' plane, belonging to the transformation $x' + i\rho' = \wp(\alpha + i\beta)$ are found by use of the addition formula for $\wp$. They may be put in Cayley's form

68) $$\left[(x'-\lambda)^2 + \rho'^2 - 3\lambda^2 + \frac{g_2}{4}\right]^2 = (4\lambda^3 - g_2\lambda - g_3)(2x' + \lambda)$$

where $\lambda = \wp(2\alpha)$ for the family α = constant or $\lambda = \wp(2i\beta)$ for the orthogonal family, and $x' + i\rho' = \wp(\alpha + i\beta)$.

The equation $(65)_b$ gives x' and ρ' in terms of x and ρ so these being placed in (68) give the meridian curves of the $x\rho$ plane. The three new arbitrary real constants A/c, B/c, D/c thus introduced, together with the two arbitrary real invariants g_2, g_3 correspond to the fact that the quartic R has five arbitrary real coefficients, which could be introduced into (68) instead of the five enumerated first. When $g_2^3 - 27g_3^2 > 0$ the roots of $(63)_c$ are all real and the four roots of the quartic are real.

Another group of transformations is associated with a quartic having some complex roots. Taking s as one of the complex roots in the solution (64) and then subjecting it to a general real homographic transformation, we may write this group of transformations in the form

69) $$z \equiv x + i\rho = A_1 + iA_2 \left[\frac{\wp(u) - e_1 + \sqrt{(e_1-e_2)(e_1-e_3)}\, e^{i\gamma}}{\wp(u) - e_1 - \sqrt{(e_1-e_2)(e_1-e_3)}\, e^{i\gamma}} \right]$$

This also contains five arbitrary real constants since e_1, e_2 and e_3 are functions of g_2 and g_3 being roots of $(63)_c$ and A_1, A_2 and γ are arbitrary real constants ($A_2 \neq 0$). When $g_2^3 - 27g_3^2 > 0$ the three roots of $(63)_c$ are real and all the roots of the quartic are complex. Adopting the convention that $e_3 < e_2 < e_1$ in this case, and also the convention, when $g_2^3 - 27g_3^2 < 0$, that e_1 is real, $e_2 = -\frac{e_1}{2} + ib$, $e_3 = -\frac{e_1}{2} - ib$ where $2b = \sqrt{3e_1^2 - g_2}$ = a positive real (the quartic having two real and two complex roots) it follows that the radical in (69), $\sqrt{(e_1-e_2)(e_1-e_3)}$, is always a positive real.

The differential equations (51) for the group of transformations (69) are

$70)_a$ $$\mathcal{U}''^{m}_{(2\alpha)} + \left[(\tfrac{1}{4} - m^2) \frac{(e_1-e_2)(e_1-e_3)}{\wp(2\alpha) - e_1} + \nu \right] \mathcal{U}^{m}_{(2\alpha)} = 0$$

$70)_b$ $$\mathcal{V}''^{m}_{(2i\rho)} + \left[(\tfrac{1}{4} - m^2) \frac{(e_1-e_2)(e_1-e_3)}{\wp(2i\rho) - e_1} + \nu \right] \mathcal{V}^{m}_{(2i\rho)} = 0$$

where $\wp(2\alpha)$, $\wp(2i\beta)$, e_1 and $(e_1-e_2)(e_1-e_3)$ are always real.
The equations of the meridian curves in the x, ρ plane belonging to the transformation (69) are readily obtained if for brevity we replace $\wp(u)$ in that equation by $x'+i\rho'$ and solve it, obtaining expressions for x' and ρ' in terms of x and ρ. The required equations then result on placing these expressions for x', ρ' in (68).

In the last example in section X a transformation which corresponds to a quartic with four complex roots is considered in detail, beginning with the transformation

71)$_a$ $$z = ia_1\, dn\left(\frac{u-2iK'}{2}\right) \quad \text{where} \quad 0 < k' = \frac{a_2}{a_1} < 1$$

a_1 and a_2 being positive reals.
If we make the quadratic substitution

71)$_b$ $$z' = -ik\frac{(z^2 - a_1a_2e^{i\delta})}{z^2 + a_1a_2e^{i\delta}} \quad \text{where } k \text{ and } \delta \text{ are real,}$$

then z' as a function of $\left(\frac{u-2iK'}{2}\right)$, is one of the group of transformations (69). This may be shown by first changing z to z' in (69) and using the relation $\wp(u) - e_1 = (e_1-e_3)\frac{cn^2(v,k)}{sn^2(v,k)}$ where $v = \sqrt{e_1-e_3}\,u$, $k' = \sqrt{\frac{e_1-e_2}{e_1-e_3}}$.

From one "separable" transformation group to another such, there is an infinitude of transformations, but it is worth repeating that the associated ordinary differential equations (51) remain the same only when the transformation is a real homographic one i e from one member to another of the same group.

The homographic or linear fractional transformation is a special case of the transformation of n^{th} order

72)$_a$ $z = \dfrac{P(z')}{Q(z')}$ where P and Q are polynomials in z'

one of order n and the other of order $n-1$ (if $n>1$), so that z' is the root of an algebraic equation of n^{th} degree whose coefficients are linear functions of z. The constant coefficients in P and Q may be so chosen (and in more than one way) as to make

72)$_b$ $du = \dfrac{dz}{\sqrt{R(z)}} = \dfrac{dz'}{\sqrt{T(z')}}$ where R and T are quartics in

their respective variables. Hence if $\left(\frac{dz}{du}\right)^2 = R(z)$ this transformation of n^{th} order makes $\left(\frac{dz'}{du}\right)^2 = T(z')$. If $R(z)$ is a real quartic and (72)$_a$ makes $T(z')$ a real quartic they both lead to a separable system of coordinates. Cayley; Treatise on Elliptic Functions page 162-280.

Landen's transformation, (and also the relation between $\wp(u)$ and the Jacobian elliptic functions), corresponds to a quadratic transformation between $z = \wp(u)$ and $z' = \sin(cu)$.

It may be noted that any transformation of the group (69) is a homographic transformation of any member of the group (65) but since A_1, A_2 and γ are real ($A_2 \neq 0$) it can never be a real homographic transformation so the expression for S is different in the two cases as shown by the differential equations (67) being different from (70).

If $R(z)$ is a given real quartic with two real and 2 complex roots let c be a real and c_1 a complex root. Then by (64) the general solution of (61) may be taken in either of the forms

$$z = f(u) = c + \frac{R'(c)}{4\left[\wp(u+u_0) - \frac{R''(c)}{24}\right]} \quad \text{where } u_0 \text{ only is arbitrary}$$

or

$$z = f(u) = c_1 + \frac{R'(c_1)}{4\left[\wp(u+v_0) - \frac{R''(c_1)}{24}\right]} \quad \text{where } v_0 \text{ only is arbitrary}$$

By taking the single arbitrary constant u_0 of the first form

as zero we are led to the group of transformations (65). By taking $v_0 = 0$ in the second we are lead to the distinct class of transformations (69). This serves to emphasise the importance of the constant of integration in integrating (61).

Equation (51) was first given by Wangerin (1878) whose results with many generalizations are given in the treatise by E. Haentzschel, Studien über die Reduction der Potentialgleichung auf gewöhnlich Differential-gleichungen. Berlin G Reimer 1893.

X Applications

1 Circular cylindrical coordinates and their inversion.

The ordinary cylindrical coordinates (x, ρ) whose values are represented by abscissa and ordinate in the x, ρ half-plane, may also be regarded as curvilinear coordinates interpreted on the z'-plane by the inversion formula $zz' = -c^2$. The expressions in terms of $x\,\rho$ for the (reduced) potential $V^m_{(x,\rho)}$ at any point in the z-half plane which has assigned values on a plane whose generator is the locus of $x = x_1 =$ constant (fig 1) is also that having assigned values on a sphere whose equation is $(x' + \frac{c^2}{2x_1})^2 + \rho'^2 = (\frac{c^2}{2x_1})^2$ (fig 2).

The potential which has given values on an endless cylinder, $\rho = \rho_1 =$ constant, is also that whose values are given on the surface generated by rotation of the circle $x'^2 + (\rho' - \frac{c^2}{2\rho_1})^2 = (\frac{c^2}{2\rho_1})^2$ about its tangent line, the x'-axis, (fig 2).

Euler's equation

$$\left(D_x^2 + D_\rho^2 + \frac{1/4 - m^2}{\rho^2}\right) V^m_{(x,\rho)} = 0$$ has solutions

$$V^m_{(x,\rho)} = \sqrt{\rho}\; e^{\pm \nu x}\, C_m(\nu\rho)$$ where ν is arbitrary

and $C_m(\nu\rho)$ any cylinder function with parameter m.

(a) Potential given on the locus $x = x_1 =$ constant.

The (reduced) potential $V^m(x,\rho)$ at any point $x\rho$ of a simple distribution with (reduced) density $\bar{\sigma}(\rho)$ on the line of the x,ρ plane, $(x = x_1\ ,\ 0 < \rho < \infty)$ is

1) $$V^m(x,\rho) = 2\int_0^\infty \bar{\sigma}(\rho_1)\, Q_{m-\frac{1}{2}}\left(1 + \frac{(x-x_1)^2 + (\rho-\rho_1)^2}{2\rho\rho_1}\right) d\rho_1$$

If ν is any positive constant the normal solutions U_ν^m of the following type vanish at $|x| =$ infinity

2)$_a$ $$U_\nu^{im}(x,\rho) = \sqrt{\rho}\, e^{-\nu(x-x_1)} J_m(\nu\rho). \qquad \text{where } x_1 \leq x \leq \infty$$

2)$_b$ $$U_\nu^{om}(x,\rho) = \sqrt{\rho}\, e^{-\nu(x_1-x)} J_m(\nu\rho). \qquad \text{where } -\infty \leq x \leq x_1$$

This potential, being continuous at $x = x_1$, and vanishing like $\rho^{m+\frac{1}{2}}$ when $\rho \to 0$, is the potential of a simple distribution at $x = x_1$ whose (reduced) density is given by

2)$_c$ $$4\pi \bar{\sigma}_\nu(\rho) = -(D_x U^{im})_{x=x_1+0} + (D_x U^{om})_{x=x_1-0} = 2\sqrt{\rho}\,\nu J_m(\nu\rho).$$

Hence $U_\nu^m(x,\rho)$ must also be given as an integral in (1) which shows that $J_m(\nu\rho)$ is a solution of the homogeneous integral equation

3) $$\int_0^\infty \sqrt{\rho_1}\, J_m(\nu\rho_1)\, Q_{m-1/2}\left(1+\frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1}\right)d\rho_1 = \frac{\pi\sqrt{\rho}}{\nu}\, e^{-\nu|x-x_1|} J_m(\nu\rho)$$

Referring to Hankel's integral representation

4) $$f(\rho) = \int_0^\infty \nu J_m(\nu\rho)\,d\nu \int_0^\infty \rho_1 f(\rho_1) J_m(\nu\rho_1)\,d\rho_1$$ it is evident

that the integral equation (3) gives the Hankel's transform for the function of ρ, $\rho^{-\frac{1}{2}} Q_{m-\frac{1}{2}}\left(1+\frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1}\right)$ in which $x-x_1$ and ρ_1 are constants. Therefore

5) $$Q_{m-1/2}\left(1+\frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1}\right) = \pi\sqrt{\rho\rho_1}\int_0^\infty e^{-\nu|x-x_1|} J_m(\nu\rho) J_m(\nu\rho_1)\,d\nu$$

which is valid for all values of $x-x_1$, ρ and ρ_1 for which $Q_{m-\frac{1}{2}}$ is finite, the integral being convergent except when $x-x_1 = \rho-\rho_1 = 0$. This representation is of course interpretable in the $x'\rho'$ plane of fig 2.

Similarly the potential

6) $$U^m(x,\rho) = \sqrt{\rho}\int_0^\infty \nu e^{-\nu|x-x_1|} J_m(\nu\rho)\,d\nu \int_0^\infty \sqrt{\rho_1}\, F(\rho_1) J_m(\nu\rho_1)\,d\rho_1$$ is one which

has assigned values on the plane $x = x_1$ or on the sphere $(x'+\frac{c^2}{2x_1})^2 + \rho'^2 = (\frac{c^2}{2x_1})^2$.

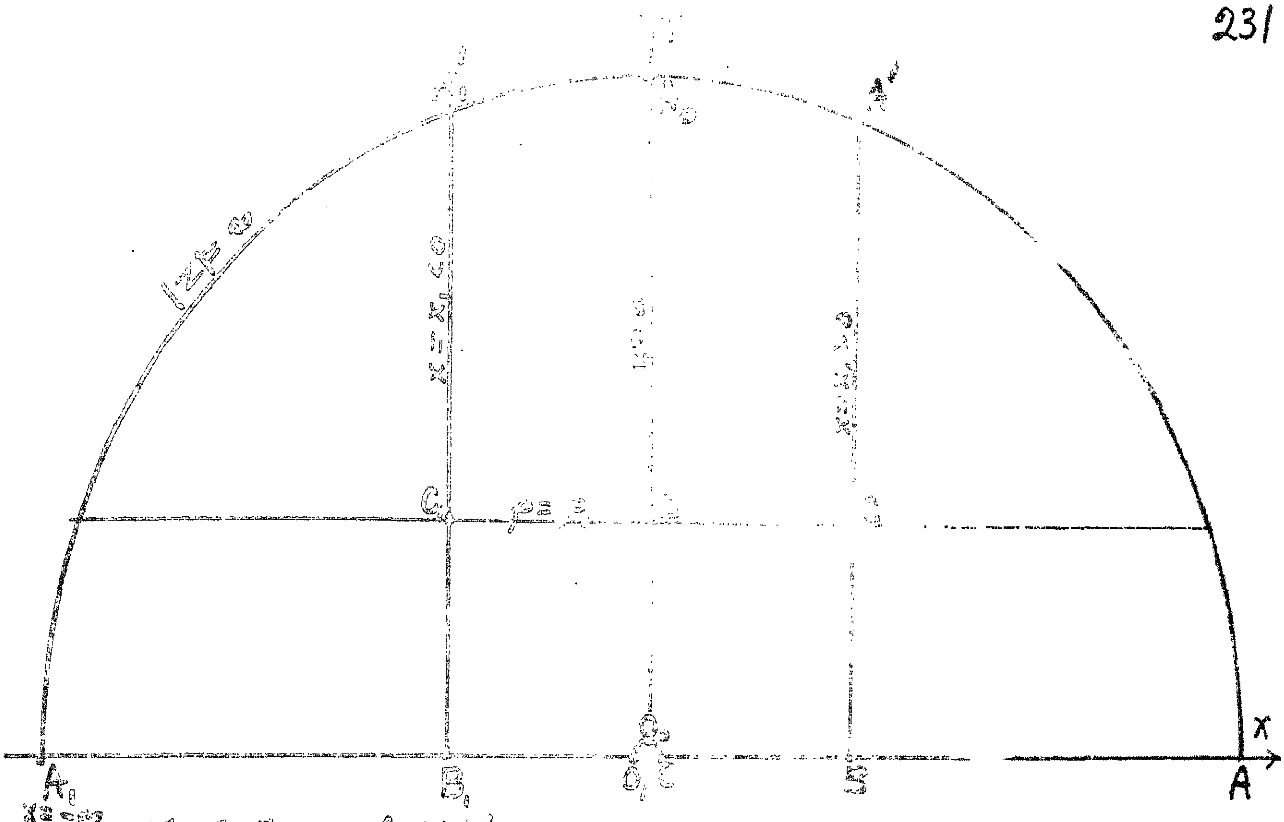


Fig 1 The x, ρ half-plane

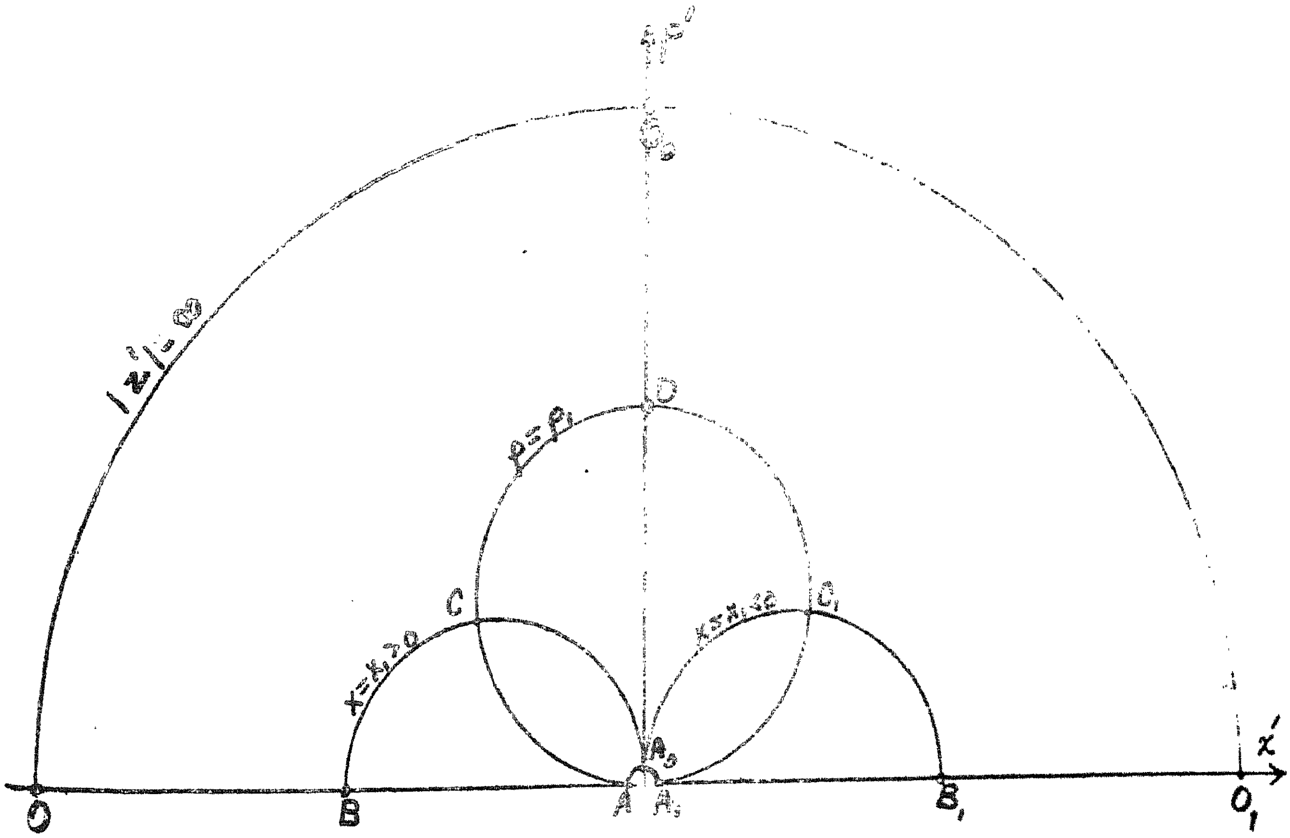


Fig 2 The x', ρ' half-planes $\quad z z' = -c^2$

(b). Potential given on the locus $\rho = \rho_1 =$ constant.

The (reduced) potential $\overset{m}{U}(x,\rho)$ of a simple distribution with (reduced) density $\bar{\sigma}(x)$ on the endless cylinder $\rho = \rho_1$ is given by

7) $$\overset{m}{U}(x,\rho) = 2\int_{-\infty}^{\infty} \bar{\sigma}(x_1)\, Q_{m-1/2}\left(1 + \frac{(x-x_1)^2 + (\rho-\rho_1)^2}{2\rho\rho_1}\right) dx_1$$

If $\overset{m}{H_1}$ denotes the first Hankel's function

8) $$J_m(i\nu\rho)\, \overset{m}{H_1}{}'(i\nu\rho) - J_m'(i\nu\rho)\, \overset{m}{H_1}(i\nu\rho) = \frac{2}{\pi\nu\rho}. \qquad (i = e^{\frac{i\pi}{2}})$$

If ν is a large positive real

9)$_q$ $$\overset{m}{H_1}(i\nu\rho) \sim \sqrt{\frac{2}{\pi i\nu\rho}}\, e^{-\nu\rho - (m+\frac{1}{2})\frac{i\pi}{2}}$$

9)$_t$ $$J_m(i\nu\rho) \sim \frac{1}{\sqrt{2\pi i\nu\rho}}\, e^{\nu\rho + (m+\frac{1}{2})\frac{i\pi}{2}}$$

The potential integral (7)$_t$ converges for any finite point (x,ρ), if $\sigma(x_1)$ becomes infinite like $|x_1|^{\delta-1}$ when $x_1 \to \pm\infty$ provided that $\delta < m + 1/2$.

Normal solutions of the form

$10)_a \quad \bar{U}^{i,m}_{\nu}(x,\rho) = \sqrt{\rho\rho_1}\, \bar{J}_m(i\nu\rho)\, i\, H^m_1(i\nu\rho_1)\, e^{i\nu x}$ where $0 \leq \rho \leq \rho_1$

$10)_b \quad \bar{U}^{0,m}_{\nu}(x,\rho) = \sqrt{\rho\rho_1}\, \bar{J}_m(i\nu\rho_1)\, i\, H^m_1(i\nu\rho)\, e^{i\nu x}$ where $\rho_1 \leq \rho \leq \infty$

being continuous at $\rho = \rho_1$ and vanishing with ρ like $\rho^{m+\frac{1}{2}}$, represent the potential of a simple distribution on the endless cylinder $\rho = \rho_1$ whose density $\bar{\sigma}_\nu(x)$ is found by use of (8),

$$10)_c \quad 4\pi\bar{\sigma}_\nu(x) = -\left(\frac{\partial}{\partial\rho} U^{0m}\right)_{\rho=\rho_1+0} + \left(\frac{\partial}{\partial\rho} U^{im}\right)_{\rho=\rho_1-0} = \frac{2e^{i\nu x}}{\pi}.$$

Hence, considering ν positive, these expressions, used in (7) shew that $e^{i\nu x}$ is a solution of the integral equation

$$11) \quad \frac{1}{\pi}\int_{-\infty}^{\infty} e^{i\nu x_1} Q_{m-\frac{1}{2}}\left(1 + \frac{(x-x_1)^2 + (\rho-\rho_1)^2}{2\rho\rho_1}\right) dx_1 = \pi\sqrt{\rho\rho_1}\, J_m(i\nu\rho)\, i\, H^m_1(i\nu\rho)\, e^{i\nu x}$$

where $i\nu\rho = \nu\rho e^{\frac{i\pi}{2}}$ and $\nu\rho$ is a positive real $\quad$ $0 \leq \rho \leq \rho_1$

so that $J_m(i\nu\rho)\, i\, H^m_1(i\nu\rho)$ is real and (11) breaks into

$$11)_a \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \cos\nu x_1\, Q_{m-\frac{1}{2}}\left(1 + \frac{(x-x_1)^2 + (\rho-\rho_1)^2}{2\rho\rho_1}\right) dx_1 = \pi\sqrt{\rho\rho_1}\, J_m(i\nu\rho)\, i\, H^m_1(i\nu\rho)\cos\nu x$$

$$11)_b \quad \frac{1}{\pi}\int_{-\infty}^{\infty} \sin\nu x_1\, Q_{m-\frac{1}{2}}\left(1 + \frac{(x-x_1)^2 + (\rho-\rho_1)^2}{2\rho\rho_1}\right) dx_1 = \pi\sqrt{\rho\rho_1}\, J_m(i\nu\rho)\, i\, H^m_1(i\nu\rho)\sin\nu x$$

Developing $Q_{m-\frac{1}{2}}$ as a function of x in a Fourier's integral, gives by use of (11)a and (11)b.

$$12)\quad Q_{m-1/2}\left(1+\frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1}\right)=i\pi\sqrt{\rho\rho_1}\int_0^\infty J_m(i\nu\rho)\,H_1^m(i\nu\rho_1)\cos\nu(x-x_1)\,d\nu$$

when $0\leq\rho\leq\rho_1$

The external and internal potentials which reduce to $F(x)$ on the cylinder $\rho=\rho_1$ are

$$13)_a\quad U^{om}_{(x,\rho)}=\frac{1}{\pi}\sqrt{\frac{\rho}{\rho_1}}\int_0^\infty\frac{H_1^m(i\nu\rho)}{H_1^m(i\nu\rho_1)}d\nu\int_{-\infty}^\infty F(x_1)\cos\nu(x-x_1)\,dx_1 \quad \text{where } \rho_1\leq\rho\leq\infty$$

and

$$13)_b\quad U^{im}_{(x,\rho)}=\frac{1}{\pi}\sqrt{\frac{\rho}{\rho_1}}\int_0^\infty\frac{J_m(i\nu\rho)}{J_m(i\nu\rho_1)}d\nu\int_{-\infty}^\infty F(x_1)\cos\nu(x-x_1)\,dx_1 \quad \text{where } 0\leq\rho\leq\rho_1$$

Interpreted by fig 2 there are potentials which reduce to assigned values on the "doughnut" without a hole, whose generator, in the $x'\rho'$ plane, is the circle $x'^2+\left(\rho'-\frac{c^2}{2\rho_1}\right)^2=\left(\frac{c^2}{2\rho_1}\right)^2$.

2 Polar and Dipolar Coordinates

Let $\omega = \alpha + i\beta$ where $0 < \alpha < \pi$ and $-\infty < \beta < \infty$.

On this endless strip of the w-plane (fig 2) the z half plane (fig 1) is represented by the equation

1) $z \equiv x + i\rho \equiv r e^{i\alpha} = e^{i\omega} = e^{-\beta} \cdot e^{i\alpha}$ so that $r \equiv e^{-\beta}$ and α are plane polar coordinates.

$$x = r\cos\alpha = e^{-\beta}\cos\alpha, \quad \rho = r\sin\alpha = e^{-\beta}\sin\alpha$$

$$\sqrt{dx^2 + d\rho^2} = \frac{\sqrt{d\alpha^2 + d\beta^2}}{h} \quad \text{where } h(\beta) = e^{\beta} = \frac{1}{r}$$

$$S(\alpha,\beta) = \frac{1}{\rho^2 h^2} = \frac{1}{\sin^2\alpha}$$

α, β are Dipolar coordinates for the z'-half-plane where

$$z = \frac{z' - c}{z' + c} \quad (c \text{ real})$$

The z'-half-plane is represented on the same strip fig 2 as shown by fig 1' by the equation

1)' $z' = ic\cot\frac{\omega}{2} = c\,\frac{1 + e^{i\omega}}{1 - e^{i\omega}}$ or $\begin{cases} x' = \dfrac{c\sinh\beta}{\cosh\beta - \cos\alpha} \\ \rho' = \dfrac{c\sin\alpha}{\cosh\beta - \cos\alpha} \end{cases}$

or

$$x'^2 + (\rho' - c\cot\alpha)^2 = \left(\frac{c}{\sin\alpha}\right)^2 \text{ and } (x' - c\coth\beta)^2 + \rho'^2 = \left(\frac{c}{\sinh\beta}\right)^2$$

$$\sqrt{dx'^2 + d\rho'^2} = \frac{\sqrt{d\alpha^2 + d\beta^2}}{h'} \quad \text{where } h'(\alpha,\beta) = \frac{\cosh\beta - \cos\alpha}{c}$$

$$S(\alpha,\beta) = \frac{1}{\rho'^2 h'^2} = \frac{1}{\sin^2\alpha}$$

Euler's equation,

$$\left[D_x^2 + D_\rho^2 + \frac{\frac{1}{4} - m^2}{\rho^2}\right] U^m = 0$$, is transformed in either case

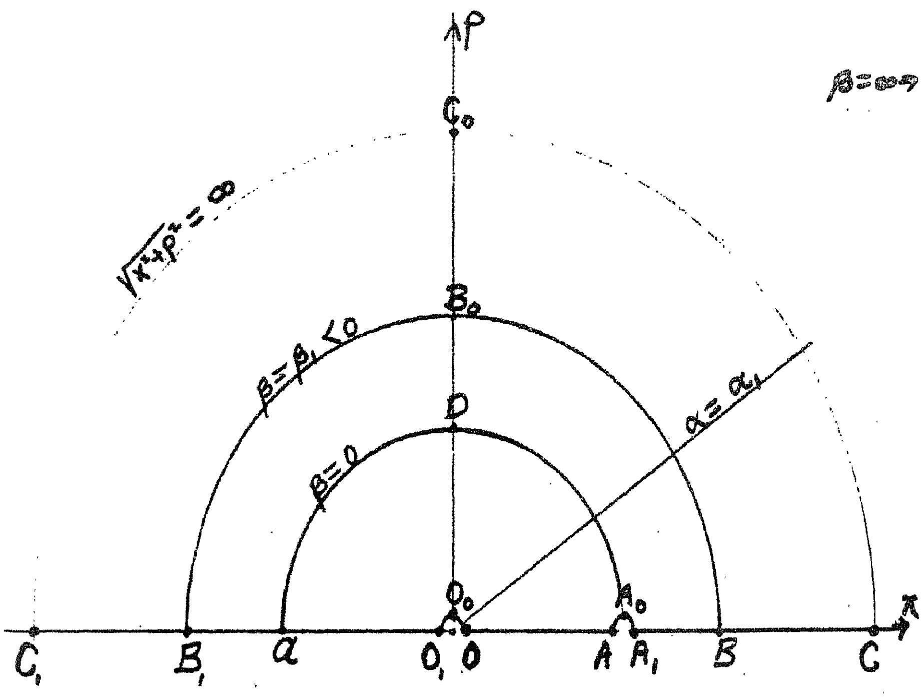


Fig 1 The z half-plane

$$ZZ'=-c^2$$

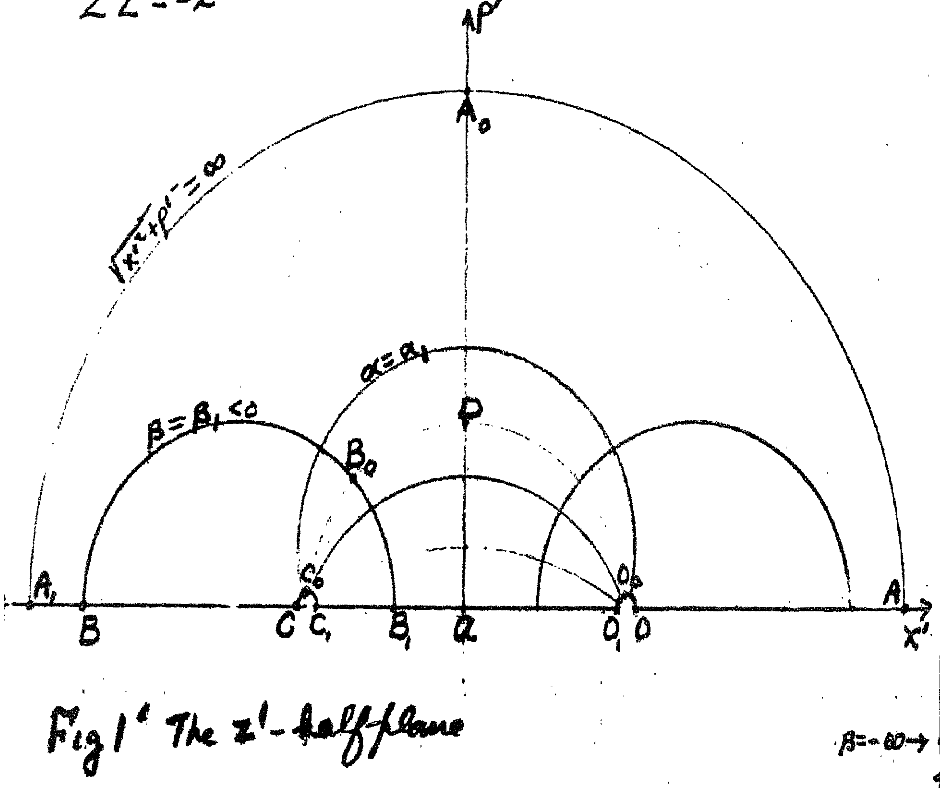


Fig 1' The z'-half-plane

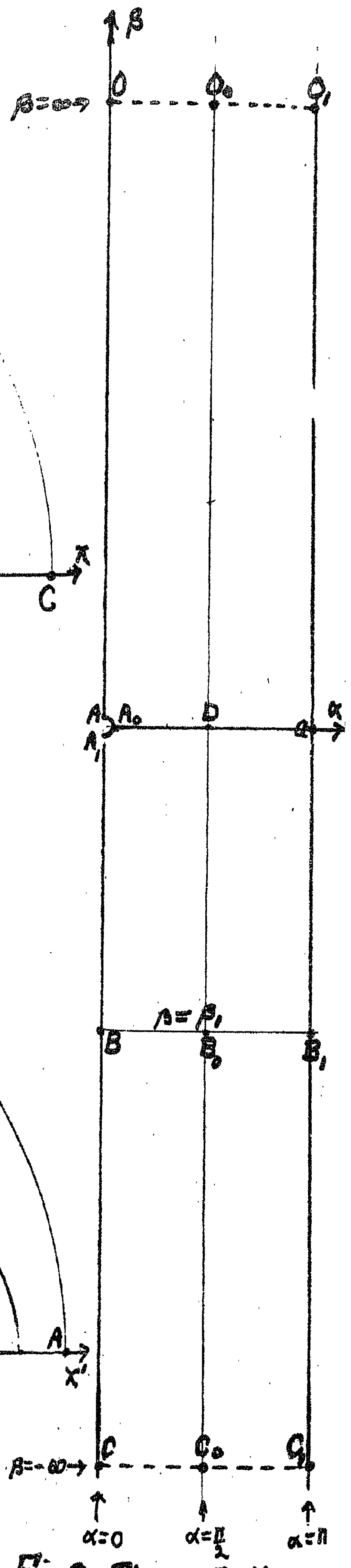


Fig 2 The w-strip

into

2) $\left(D_\alpha^2 + D_\beta^2 + \frac{\frac{1}{4}-m^2}{\sin^2\alpha}\right)U^m = 0$, which has solutions of the form $U^m = u^m(\alpha)\, v^m(\beta)$ where

3)$_a$ $u''^m(\alpha) + \left[\frac{1/4 - m^2}{\sin^2\alpha} + \nu^2\right] u^m(\alpha) = 0$

and

3)$_b$ $v''^m(\beta) - \nu^2 v^m(\beta) = 0$

Letting $u^m = \sqrt{\sin\alpha}\; y$, eq (3) becomes

4) $\frac{d^2y}{d\alpha^2} + \cot\alpha \frac{dy}{d\alpha} + \left(\nu - \frac{1}{4} - \frac{m^2}{\sin^2\alpha}\right) y = 0$

or

5) $\frac{d}{d\xi}\left[(1-\xi^2)\frac{dy}{d\xi}\right] + \left[\nu^2 - \frac{1}{4} - \frac{m^2}{1-\xi^2}\right] y = 0$ where $\xi = \cos\alpha$

Hence, taking $\nu - \frac{1}{2} = n$, the eq (2) has solutions of the form

6)$_a$ $U = \sqrt{\sin\alpha}\; T_n^m(\cos\alpha)\left[A e^{(n+\frac{1}{2})\beta} + B e^{-(n+\frac{1}{2})\beta}\right]$

or replacing ν by $i\nu$

6)$_b$ $U = \sqrt{\sin\alpha}\,[A\cos\nu\beta + B\sin\nu\beta]\left[C\, T^m_{i\nu-1/2}(\cos\alpha) - D\, T^m_{i\nu-1/2}(-\cos\alpha)\right]$

The condition IX (13) for no sources on the x axis is

7)$_a$ $U \to 0$ like $\sin\alpha^{m+\frac{1}{2}}$ when $\alpha \to 0$ or $\alpha \to \pi$ $(\rho \to 0)$

And by IX (13)$_a$ for no sources at infinity

7)$_b$ $U \to 0$ like $e^{(m+\frac{1}{2})\beta}$ when $\beta \to -\infty$ (i.e. like $r^{-(m+\frac{1}{2})}$ as $r = |z| \to \infty$)

Similarly for no sources at the point O, U must be finite when $\beta \to +\infty$.

(a) Potential given on a sphere, $\beta = \beta_1$, $0 < \alpha < \pi$.
We require solutions with no sources on the x axis where $x > 0$, $\cos\alpha = +1$ and none on the other half $x < 0$, $\cos\alpha = -1$. It was shown in VI after eq(44)' that when m is a given non-negative integer the only solutions of eq(5) which remain finite when $\cos\alpha = +1$ and also when $\cos\alpha = -1$, are $T_n^m(\cos\alpha)$ where n is an integer, $n \geq m$. Consequently the harmonics which are internal and external with respect to the semi-circle of fig 1 (whose equation is $\beta = \beta_1$), and which are continuous there, are of the form

8)$_a$ $$U_n^{im}(\alpha,\beta) = e^{(n+\frac{1}{2})(\beta_1-\beta)} \cdot \mathcal{U}_n^m(\alpha) \qquad \text{where } \beta_1 \leq \beta \leq +\infty$$

8)$_b$ $$U_n^{om}(\alpha,\beta) = e^{(n+\frac{1}{2})(\beta-\beta_1)} \mathcal{U}_n^m(\alpha) \qquad \text{where } -\infty \leq \beta \leq \beta_1$$

This is the potential of a simple distribution on the semi-circle $\beta = \beta_1$ whose (reduced) density is given by

8)$_c$ $$\frac{4\pi \bar{\sigma}_n(\alpha)}{h(\alpha,\beta_1)} = (2n+1)\, \mathcal{U}_n^m(\alpha) \qquad \text{where}$$

9) $$\begin{cases} \mathcal{U}_n^m(\alpha) \equiv C_n^m \sqrt{\sin\alpha}\; T_n^m(\cos\alpha) \quad \text{and} \quad C_n^m = \sqrt{(n+\frac{1}{2})\frac{(n-m)!}{(n+m)!}} \\ \int_0^\pi \mathcal{U}_{n_1}^m(\alpha)\, \mathcal{U}_{n_2}^m(\alpha)\, d\alpha = \delta_{n_1,n_2} \end{cases}$$

The development of an arbitrary function $f(\alpha)$ is

10)$_a$ $$f(\alpha)=\sum_{n=m}^{\infty} \mathcal{U}_n^m(\alpha)\int_0^\pi f(\alpha_1)\,\mathcal{U}_n^m(\alpha_1)\,d\alpha_1 \quad , \qquad \text{for } 0<\alpha<\pi,$$

that is,

10)$_b$ $$f(\alpha) = \sqrt{\sin\alpha}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}\,T_n^m(\cos\alpha)\int_0^\pi f(\alpha_1)\sqrt{\sin\alpha_1}\,T_n^m(\cos\alpha_1)\,d\alpha_1$$

Hence the (reduced) potential which has given values, $f(\alpha)$, on the sphere $\beta=\beta_1$ is

11) $$U^m(\alpha,\beta)=\sqrt{\sin\alpha}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}\,e^{(n+\frac{1}{2})(\beta-\beta_1)}\,T_n^m(\cos\alpha)\int_0^\pi f(\alpha_1)\sqrt{\sin\alpha_1}\,T_n^m(\cos\alpha_1)\,d\alpha_1$$

where $-\infty<\beta\leq\beta_1$, these being interchanged in the other case. Interpreting this on the $x\rho$ half-plane as in fig 1. $e^{-\beta}=n$ and $e^{-\beta_1}=n_1$. Interpreted on the $x'\rho'$ plane we place $e^{-\beta}=\frac{R_1}{R_2}=\left|\frac{z'-c}{z'+c}\right|$.

Since $\frac{\sigma}{h}$ is invariant, the potentials $8)_a$, $8)_b$ and their density $8)_c$ are in invariant form. Both $8)_a$ and $8)_b$ must be included in the potential integral

12) $$U^m(\alpha,\beta)=2\int\bar{\sigma}\,Q_{m-\frac{1}{2}}(\omega)\,ds_1 = 2\int_0^\pi\frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1,\beta_1)}\,Q_{m-\frac{1}{2}}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right)d\alpha_1$$

where

13) $$g(\alpha,\beta;\alpha_1,\beta_1)=\frac{\cosh(\beta-\beta_1)-\cos\alpha\cos\alpha_1}{\sin\alpha\sin\alpha_1}$$

Placing in (12) the expression $(8)_e$ for $\bar{\sigma}/h$ and equating U to U^i or U^o shows that $T_n^m(\cos\alpha)$ is a solution of the homogenous integral equation

$$14)\quad \int_0^\pi \sqrt{\sin\alpha_1}\, T_n^m(\cos\alpha_1)\, Q_{m-1/2}\big(g(\alpha,\beta;\alpha_1,\beta_1)\big)\, d\alpha_1 = \frac{\pi e^{(n+\frac{1}{2})(\beta_1-\beta)}\cdot\sqrt{\sin\alpha}}{(n+1/2)}\, T_n^m(\cos\alpha) \quad \text{if } \beta_1 \le \beta \le \infty$$

$$= \frac{\pi e^{(n+\frac{1}{2})(\beta-\beta_1)}\cdot\sqrt{\sin\alpha}}{(n+1/2)}\, T_n^m(\cos\alpha) \quad \text{if } -\infty \le \beta \le \beta_1$$

Considering $Q_{m-1/2}(g)$ as a function of α_1 in which α, β and β_1 are constants, it may be developed in a normal series of type (10) in which the Fourier coefficients are given by this integral equation. The result is

$$15)\quad Q_{m-1/2}\left(\frac{\cosh(\beta-\beta_1)-\cos\alpha\cos\alpha_1}{\sin\alpha\sin\alpha_1}\right) = \pi\sqrt{\sin\alpha\sin\alpha_1}\sum_{n=m}^{\infty}\frac{(n-m)!}{(n+m)!}\, e^{-(n+\frac{1}{2})(\beta-\beta_1)}\, T_n^m(\cos\alpha)\, T_n^m(\cos\alpha_1).$$

where $\beta_1 \le \beta \le \infty$.

(b) Potential given on a Cone or Spindle ($\alpha = \alpha_1, -\infty < \beta < \infty$).

The (reduced) potential $U^{m}_{(\alpha,\beta)}$ of a simple distribution with (reduced) density $\bar{\sigma}(\ )$ on the locus $\alpha = \alpha_1$ (Cone or Spindle is given everywhere by the integral

16) $$U^{m}_{(\alpha,\beta)} = 2\int_{-\infty}^{\infty} \frac{\bar{\sigma}(\beta)}{h(\alpha_1,\beta_1)} Q_{m-1/2}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right) d\beta_1$$

To construct harmonics with solutions of type (6)$_b$, the condition for no sources on the x axis, requires the use of $T^{m}_{i\nu-1/2}(\cos\alpha)$ in that region which includes the locus α and $T^{m}_{i\nu-1/2}(-\cos\alpha)$ in the other region where α may take the value π. Hence consider the continuous solution

17)$_a$ $U^{im}_{(\alpha,\beta)} = (A_\nu \cos\nu\beta + B_\nu \sin\nu\beta)\sqrt{\sin\alpha \sin\alpha_1}\, T^{m}_{i\nu-1/2}(\cos\alpha)\, T^{m}_{i\nu-1/2}(-\cos\alpha_1)$ where $0 \leq \alpha \leq \alpha_1$

(inside the cone or outside the spindle) and

17)$_b$ $U^{0m}_{(\alpha,\beta)} = (A_\nu \cos\nu\beta + B_\nu \sin\nu\beta)\sqrt{\sin\alpha \sin\alpha_1}\, T^{m}_{i\nu-1/2}(-\cos\alpha)\, T^{m}_{i\nu-1/2}(\cos\alpha_1)$ where $\alpha_1 \leq \alpha \leq \pi$

These satisfy the condition for no charges on the x axis; independently of the parameter ν. If is real, this will be the potential of a simple charge of (reduced) density $\bar{\sigma}(\beta)$,

17)$_c$ $$4\pi \frac{\bar{\sigma}(\beta)}{h(\alpha_1)} = \frac{(A_\nu \cos\nu\beta + B_\nu \sin\nu\beta)}{\Gamma(\frac{1}{2}-m+i\nu)\,\Gamma(\frac{1}{2}-m-i\nu)} = \left(D_\alpha U^{i}\right)_{\alpha=\alpha_1-0} - \left(D_\alpha U^{0}\right)_{\alpha=\alpha_1+0}$$

This result is obtained by use of VI (26)$_a$ which

may be written

$$18)\quad \sqrt{\sin\alpha}\left[T^{m}_{i\nu-1/2}(-\cos\alpha)\, D_{\alpha}\left(\sqrt{\sin\alpha}\, T^{m}_{i\nu-1/2}(\cos\alpha)\right) - T^{m}_{i\nu-1/2}(\cos\alpha)\, D_{\alpha}\left(\sqrt{\sin\alpha}\, T^{m}_{i\nu-1/2}(-\cos\alpha)\right)\right] =$$

$$= \frac{2}{\Gamma(\frac{1}{2}-m+i\nu)\,\Gamma(\frac{1}{2}-m-i\nu)}$$

Inserting the expression $(17)_c$ for $\bar{\sigma}/h$ in the integral (16) and equating U to U^i or U^o shows that $\cos\nu\beta$ and $\sin\nu\beta$ are solutions of the homogenous integral equation

$$19)_a\quad \int_{-\infty}^{\infty} \cos\nu\beta_1\, Q_{m-1/2}(q(\alpha,\beta;\alpha_1,\beta_1))\, d\beta_1 = \pi\,\Gamma(\tfrac{1}{2}-m+i\nu)\,\Gamma(\tfrac{1}{2}-m-i\nu)\sqrt{\sin\alpha\sin\alpha_1}\; T^{m}_{i\nu-1/2}(\cos\alpha)\, T^{m}_{i\nu-1/2}(-\cos\alpha_1)\cos\nu\beta$$

and a similar equation where sines replace cosines, both equations being valid for $0 \leq \alpha \leq \alpha_1$. These integrals are the Fourier transforms of $Q_{m-1/2}(q)$ as a function of β.

Consequently

$$20)\quad Q\left(\frac{\cosh(\beta-\beta_1) - \cos\alpha\cos\alpha_1}{\sin\alpha\sin\alpha_1}\right) =$$

$$= \sqrt{\sin\alpha\sin\alpha_1}\int_{0}^{\infty} \cos\nu(\beta-\beta_1)\,\Gamma(\tfrac{1}{2}-m+i\nu)\,\Gamma(\tfrac{1}{2}-m+i\nu)\; T^{m}_{i\nu-1/2}(\cos\alpha)\; T^{m}_{i\nu-1/2}(-\cos\alpha_1)\, d\nu$$

when $0 \leq \alpha \leq \alpha_1$.

The potential which has assigned values $f(\beta)$ on the cone or spindle $\alpha = \alpha_1$ is given by

21) $$U^{m}(\alpha,\beta)=\frac{1}{\pi}\sqrt{\frac{\sin\alpha}{\sin\alpha_1}}\int_0^\infty \frac{T^{m}_{i\nu-1/2}(\cos\alpha)}{T^{m}_{i\nu-1/2}(\cos\alpha_1)}\,d\nu\int_{-\infty}^{\infty} f(\beta_1)\cos\nu(\beta-\beta_1)\,d\beta_1$$

when $0\leq\alpha\leq\alpha_1$. For the remaining region where $\alpha_1\leq\alpha\leq\pi$, $\cos\alpha$ and $\cos\alpha_1$ in this integral must be replaced by $-\cos\alpha$ and $-\cos\alpha_1$ respectively.

3 Toroidal Coordinates

The x,ρ-half-plane of fig 1 is represented on the semi-infinite strip of the w-plane $-\pi < \alpha < \pi$, $0 < \beta < \infty$, fig 2, by the equation

$$1)\quad \left\{\begin{array}{l} z = -c \cot w/2 \\ \text{or}\quad e^{iw} = \dfrac{z - ic}{z + ic} \end{array}\right\} \text{ or } \left\{\begin{array}{l} x = \dfrac{-c \sin\alpha}{\cosh\beta - \cos\alpha} \\ \rho = \dfrac{c \sinh\beta}{\cosh\beta - \cos\alpha} \end{array}\right\} \qquad c > 0$$

The family of circles, $\beta =$ constant, each of which generates a torus, belongs to the equation

$$2)_a \quad x^2 + (\rho - c \coth\beta)^2 = \left(\frac{c}{\sinh\beta}\right)^2.$$

The equation of the family of circular arcs, orthogonal to these is

$$2)_b \quad (x + c \cot\alpha)^2 + \rho^2 = \left(\frac{c}{\sin\alpha}\right)^2,$$ the locus $\alpha =$ constant being less than a semicircle, since it begins on the x axis and ends at the singular point C.

From (1) it is found that

$$3)\quad \sqrt{dx^2 + d\rho^2} = \frac{\sqrt{d\alpha^2 + d\beta^2}}{h} \quad \text{where } h(\alpha,\beta) = \frac{\cosh\beta - \cos\alpha}{c} \text{ so that}$$

$$4)\quad \frac{1}{\rho^2 h^2} = \frac{1}{\sinh^2\beta}$$

The two sides of cut $\overline{OCO'}$ of fig 1 which generates both sides of a circular disc, correspond to the

two infinite sides, $\alpha = \pi$ and $\alpha = -\pi$ of the ω-strip. The same ω-strip represents also the z' half-plane of fig 1', which is cut along a circular arc. The real homographic transformation between z and z', for which Euler's equation is invariant, leads to other systems of toroidal coordinates in which the only essential change is the positions of the corresponding cuts in the two planes.

If the inversion formula is

5)$_a$ $z' = -x_0 - \dfrac{x_0^2 + c^2}{z - x_0}$, that is $\left\{ \begin{aligned} x' &= -x_0 - \frac{(x_0^2+c^2)(x-x_0)}{(x-x_0)^2+\rho^2} \\ \rho' &= -\frac{(x_0^2+c^2)\rho}{(x-x_0)^2+\rho^2} \end{aligned} \right\}$

this becomes, on placing $x_0 = -c\cot(\alpha_0/2)$

5)$_b$ $z' = -c\cot\left(\dfrac{\omega - \alpha_0}{2}\right)$, that is, $\left\{ \begin{aligned} x' &= \frac{-c\sin(\alpha-\alpha_0)}{\cosh\beta - \cos(\alpha-\alpha_0)} \\ \rho' &= \frac{c\sinh\beta}{\cosh\beta - \cos(\alpha-\alpha_0)} \end{aligned} \right.$

whence

$$5_c \quad \left\{ \begin{aligned} & x'^2 + (\rho' - c\coth\beta)^2 = \left(\frac{c}{\sinh\beta}\right)^2 \\ & (x' + c\cot(\alpha-\alpha_0))^2 + \rho'^2 = \frac{c^2}{\sin^2(\alpha-\alpha_0)} \end{aligned} \right.$$

so that $h'(\alpha,\beta) = \dfrac{\cosh\beta - \cos(\alpha-\alpha_0)}{c}$ and

5)$_d$ $\dfrac{1}{\rho'^2 h'^2} = \dfrac{1}{\sinh^2\beta}$

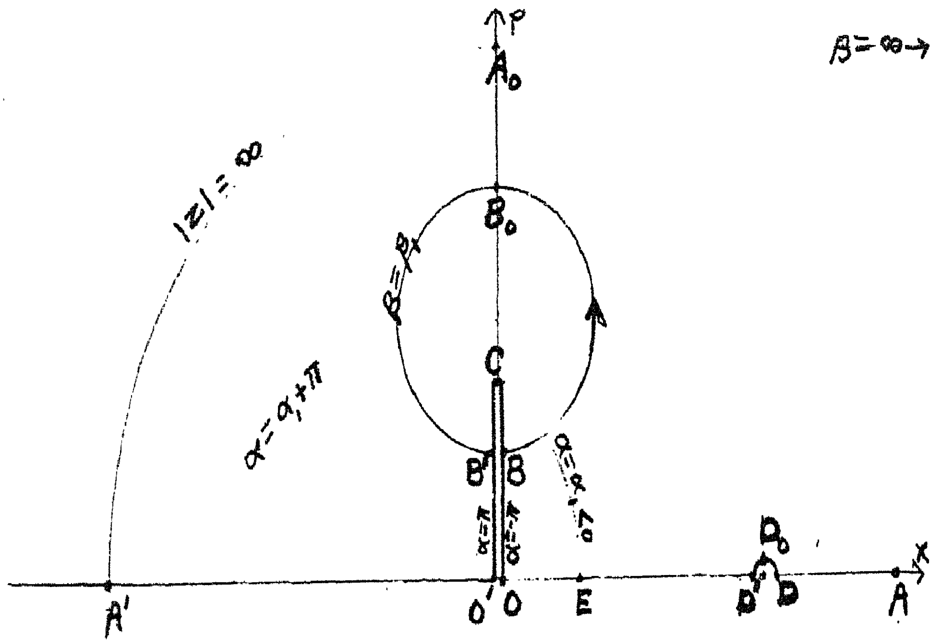


Fig 1 The z-plane

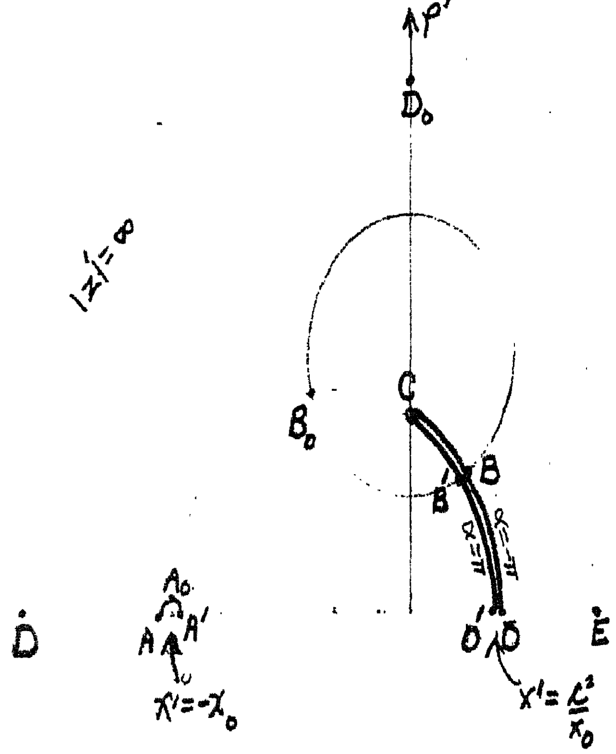


Fig 1' The z'-plane $z' = -x_0 + \dfrac{x_0^2 + c^2}{z - x_0}$

$x_0 = -c \cot\frac{\alpha_0}{2} > 0$

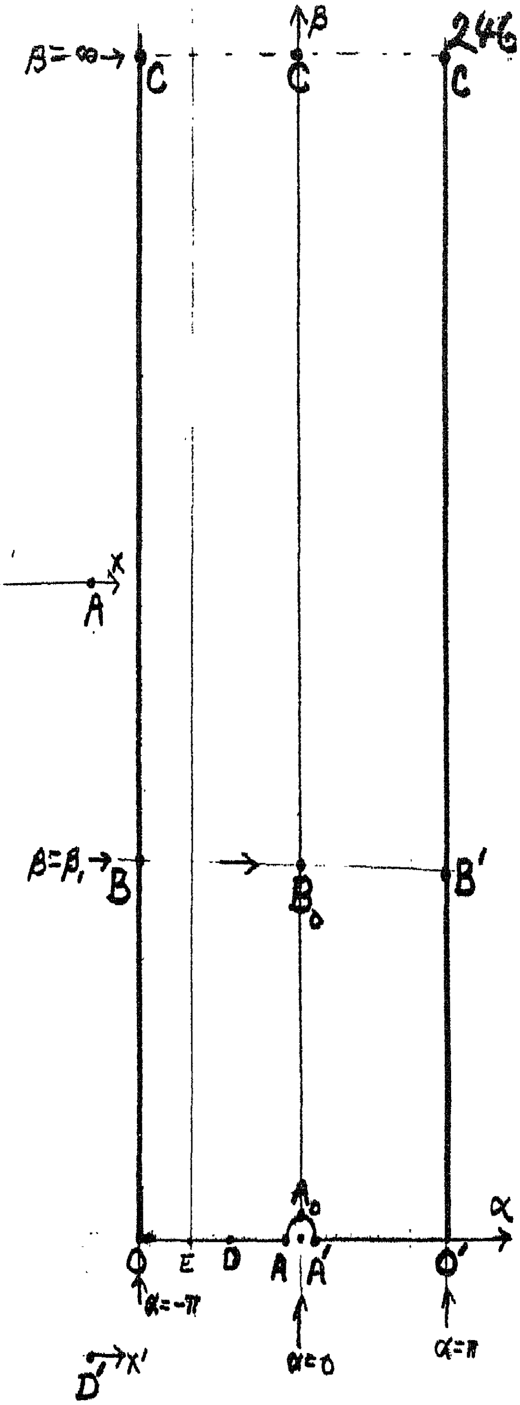


Fig 2
The w-strip

$z = -c \cot\frac{w}{2}$

$z' = -c \cot\left(\frac{w - \alpha_0}{2}\right)$

Euler's equation,

6)$_a$ $\left(D_\alpha^2 + D_\rho^2 + \frac{1/4 - m^2}{\rho^2}\right) U^m = 0$ becomes

6)$_b$ $\left(D_\alpha^2 + D_\beta^2 + \frac{1/4 - m^2}{\sinh^2\beta}\right) U^m = 0$

This has solutions of the form $U = u(\alpha)\, v(\beta)$ where

7) $u''(\alpha) + \mu^2 u(\alpha) = 0$ and

8) $v''(\beta) + \left[\frac{\frac{1}{4} - m^2}{\sinh^2\beta} - \mu^2\right] v(\beta) = 0$, or

8)$_a$ $\frac{d}{d\eta}\left[(1-\eta^2)\frac{dv}{d\eta}\right] + \left[m^2 - \frac{1}{4} - \frac{\mu^2}{1-\eta^2}\right] v = 0$ where $\eta \equiv \coth\beta$.

Hence eq (6) has solutions of the form

9)$_a$ $U^m_{(\alpha,\beta)} = (A\cos\mu\alpha + B\sin\mu\alpha)\left(C P^{\mu}_{m-1/2}(\coth\beta) + D\, Q^{\mu}_{m-1/2}(\coth\beta)\right)$

or by Whipple's formula

9)$_b$ $U^m_{(\alpha,\beta)} = (A\cos\nu\alpha + B\sin\nu\alpha)\sqrt{\sinh\beta}\left[C P^{m}_{\nu-1/2}(\cosh\beta) + D\, Q^{m}_{\nu-1/2}(\cosh\beta)\right]$

The condition for no sources on the x axis or at infinity is

10) $U^m \to 0$ like $\beta^{m+\frac{1}{2}}$ when $\beta \to 0$ for $-\pi < \alpha < \pi$ (IX (13)).

The "point at infinity", $A\, A_0\, A'$, of fig 1, corresponds to $\alpha = \beta = 0$, but any point on the line, $\beta = 0$, of fig 2 (such as D) may correspond to infinity in some inverse plane as in fig 1'.

(a). Potential given on a toroid, $\beta = \beta_1$.

The circle $\beta = \beta_1$ of fig 1, which is a meridian section of the toroid, is always indented by the cut. Internal and external harmonics which represent neither simple nor double distributions at the cut must be periodic functions of α with period 2π, since the potential and also its derivatives must regain their initial values when a circuit is described around the singular point C. Hence the solutions of form (9)$_a$ must have the real integers as the eigen-values of the parameter μ.

By VI (45) and (56)$_b$ we find for $0 < \beta < \infty$

$$11)_a \quad \overset{n}{\underset{m-1/2}{P}}(\coth\beta) = \frac{\Gamma(\frac{1}{2}+m+n)}{\Gamma(\frac{1}{2}+m-n)} \frac{(1-e^{-2\beta})^{\frac{1}{2}-m} e^{-n\beta}}{n!} F(\tfrac{1}{2}-m, \tfrac{1}{2}-m+n, n+1; e^{-2\beta})$$

$$11)_b \quad \overset{n}{\underset{m-1/2}{Q}}(\coth\beta) = \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}+m+n)}{n!} \left(\frac{\tanh\beta}{2}\right)^{m+\frac{1}{2}} \operatorname{sech}^n\beta\, F(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{n}{2}, \tfrac{3}{4}+\tfrac{m}{2}+\tfrac{n}{2}, n+1; \tanh^2\beta)$$

and by VI (7)

$$11)_c \quad \overset{n}{\underset{m-1/2}{P}}(\coth\beta)\, D_\beta \overset{n}{\underset{m-1/2}{Q}}(\coth\beta) - \overset{n}{\underset{m-1/2}{Q}}(\coth\beta)\, D_\beta \overset{n}{\underset{m-1/2}{P}}(\coth\beta) = (-1)^n \frac{\Gamma(\frac{1}{2}+m+n)}{\Gamma(\frac{1}{2}+m-n)}$$

The internal and external harmonics with respect to the torus $\beta=\beta_1$ must be of the form;

12)$_a$ $\overset{i\,m}{U_n}(\alpha,\beta) = (A_n \cos n\alpha + B_n \sin n\alpha)\, P^n_{m-1/2}(\coth\beta)\, Q^n_{m-1/2}(\coth\beta_1)$ when $\beta_1 \leq \beta < \infty$

12)$_b$ $\overset{o\,m}{U_n}(\alpha,\beta) = (A_n \cos n\alpha + B_n \sin n\alpha)\, P^n_{m-1/2}(\coth\beta_1)\, Q^n_{m-1/2}(\coth\beta)$ when $0 \leq \beta \leq \beta_1$

This potential, being continuous must be that of a simple distribution on the torus with density given by

12)$_c$ $\dfrac{4\pi\bar{\sigma}_n(\alpha)}{h(\alpha,\beta_1)} = (-1)^n \dfrac{\Gamma(\frac{1}{2}+m+n)}{\Gamma(\frac{1}{2}+m-n)}(A_n \cos n\alpha + B_n \sin n\alpha)$ which is

found by the general formula

13) $\dfrac{4\pi\bar{\sigma}(\alpha)}{h(\alpha,\beta)} = -(D_\beta U)_{\beta=\beta_1+0} + (D_\beta U)_{\beta=\beta_1-0}$ together with (11)$_c$.

Both forms of (12) must be included in the potential integral

14) $\overset{m}{U}(\alpha,\beta) = 2\displaystyle\int_{-\pi}^{\pi} \frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1,\beta_1)}\, Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))\, d\alpha_1$ where

15) $g(\alpha,\beta;\alpha_1,\beta_1) \equiv \dfrac{\cosh\beta\cosh\beta_1 - \cos(\alpha-\alpha_1)}{\sinh\beta\sinh\beta_1}$

Using the expression 12)$_c$ for $\bar{\sigma}/h$ in the integral (14) and equating U to U^i or U^o gives the following integral

equation with $Q_{m-\frac{1}{2}}$ as nucleus.

$$16)\quad \frac{1}{\pi}\int_{-\pi}^{\pi} \cos n\alpha' \, Q_{-m-1/2}(\, g(\alpha,\beta;\alpha',\beta_1)\, d\alpha' = \frac{2(-1)^n \Gamma(\frac{1}{2}+m-n)}{\Gamma(\frac{1}{2}+m+n)} \cos n\alpha \, P^n_{m-1/2}(\coth\beta_1)\, Q^n_{m-1/2}(\coth\beta)$$

and a similar equation where cosines are replaced by sines both equations being valid when $0 \leq \beta \leq \beta_1$. In the other alternative we merely interchange β and β_1.

These integrals are the coefficients of the Fourier's series which represents $Q_{m-1/2}(g)$ as a function of α.

Therefore (if $\epsilon_0 = \frac{1}{2}$, and $\epsilon_n = 1$ for $n \neq 0$)

$$17)\quad Q_{m-1/2}\left(\frac{\cosh\beta\cosh\beta_1 - \cos(\alpha-\alpha_1)}{\sinh\beta\sinh\beta_1}\right) =$$

$$= 2\sum_{n=0}^{\infty} \epsilon_n (-1)^n \frac{\Gamma(\frac{1}{2}+m-n)}{\Gamma(\frac{1}{2}+m+n)} P^n_{m-1/2}(\coth\beta_1)\, Q^n_{m-1/2}(\coth\beta) \cos n(\alpha-\alpha_1)$$

which holds for all values of α an α_1 but is restricted to the case $0 \leq \beta \leq \beta_1$, that is to the case where (α,β) is a point outside the torus $\beta=\beta_1$.

It is easy to put (17) in the form of a contour integral which exhibits the fact that $Q_{m-1/2}(g)$ is a symmetric function of β and β_1, but this is done at the expense of changing the criterion from the β's to the α's. This integral is

$$18)\quad Q_{m-1/2}\big(g(\alpha,\beta;\alpha_1,\beta_1)\big)=\frac{1}{\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\cos\mu\left[\pi-|\alpha-\alpha_1|\right]}{\cos^2\mu\pi}\,\frac{\Gamma(\frac{1}{2}+m-\mu)}{\Gamma(\frac{1}{2}+m+\mu)}\,Q^{\mu}_{m-1/2}(\coth\beta)\cdot Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\mu$$

where μ_1 may be taken in the interval $-m-\frac{1}{2}<\mu_1<m+\frac{1}{2}$, since the integrand contains no poles in this strip of the complex μ plane, $Q^{\mu}_{m-1/2}/(\cos\mu\pi\,\Gamma(\frac{1}{2}+m+\mu))$ being an integral function of μ. It will be seen that this integral represents $Q_{m-1/2}(g)$ for all positive values of β and β_1 and it is symmetric in these variables. However it is not a single valued function of $\alpha-\alpha_1$, because of the absolute value sign, $|\alpha-\alpha_1|$, so that (18) actually is a condensed form of two cases $1°$, $\alpha\le\alpha_1$ and $2°$, $\alpha\ge\alpha_1$.

To show that both forms of (18) cover both cases of (17) we may express one of the Q functions, say the second, in terms of P functions by the fundamental formula VI (6),

$$Q^{\mu}_{m-1/2}(\coth\beta_1)=-\frac{\pi}{2}\cot\mu\pi\left[P^{\mu}_{m-1/2}(\coth\beta_1)-\frac{\Gamma(\frac{1}{2}+m+\mu)}{\Gamma(\frac{1}{2}+m-\mu)}P^{-\mu}_{m-1/2}(\coth\beta_1)\right]$$

Before inserting this expression in the integral (18) we move the path to $\mu_1=\epsilon$ where ϵ is small and positive. Then move the path of that part of the integral containing $P^{-\mu}$ to $\mu_1=-\epsilon$ and then recover the original path $\mu_1=+\epsilon$, by changing

the sign of the variable of integration. The eq (18) is thus converted into

$$(18)' \quad Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1)) = P_{m-1/2}(\coth\beta_1)\, Q_{m-1/2}(\coth\beta)$$

$$-\frac{1}{2i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{\cos\mu(\pi-|\alpha-\alpha_1|)\ \Gamma(\frac{1}{2}+m-\mu)}{\sin\mu\pi\ \cos\mu\pi\ \Gamma(\frac{1}{2}+m+\mu)} P^{\mu}_{m-1/2}(\coth\beta_1)\, Q^{\mu}_{m-1/2}(\coth\beta)\, d\mu$$

In the same way the form with β and β_1 interchanged is found. The first term on the right comes from the pole at $\mu=0$, which was passed over in the movement of the path. The integral (18)' was obtained by using the identity $\Gamma(\frac{1}{2}+m-\mu)\, Q^{\mu}_{m-1/2} \equiv \Gamma(\frac{1}{2}+m+\mu)\, Q^{-\mu}_{m-1/2}$ of VI $(9)_a$.

If we now consider $\beta \leq \beta_1$, the path of the integral (18)' may be closed by adding an infinite semicircle on the right, and, evaluating the integral for the poles of $1/\sin\mu\pi$ thus enclosed, we obtain the series (17). To justify this closure we have, on this infinite semicircle

$$\Gamma(\tfrac{1}{2}+m-\mu)\, P^{\mu}_{m-1/2}(\coth\beta_1) \sim \left(\frac{\coth\beta_1-1}{\coth\beta_1+1}\right)^{\frac{\mu}{2}} \mu^{m-\frac{1}{2}}$$

$$\frac{Q^{\mu}_{m-1/2}(\coth\beta)}{\cos\mu\pi\ \Gamma(\frac{1}{2}+m+\mu)} \sim \frac{1}{2\,\mu^{m+\frac{1}{2}}}\left(\frac{\coth\beta-1}{\coth\beta+1}\right)^{-\frac{\mu}{2}}$$

$$\frac{\cos\mu(\pi-|\alpha-\alpha_1|)}{\sin\mu\pi} \sim 0 \quad \text{if} \quad \alpha-\alpha_1 \neq \pi$$

$$\sim \mp i \quad \text{if} \quad \alpha-\alpha_1 = \pi$$

The following special case of (18), $\alpha_1 = \pm\pi$ is valid everywhere

$$19)^{\beta}\quad Q_{m-\frac{1}{2}}\left(g(\alpha,\beta;\pi,\beta_1)\right) = \frac{1}{\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\cos\mu\alpha\,\Gamma(\frac{1}{2}+m-\mu)}{\cos^2\mu\pi\,\Gamma(\frac{1}{2}+m+\mu)} Q^{\mu}_{m-1/2}(\coth\beta)\, Q^{\mu}_{m-1/2}(\coth\beta_1)\, d\mu$$

$$-m-\tfrac{1}{2} < \mu_1 < m+\tfrac{1}{2}$$

If a (reduced) potential V^m has assigned values $f(\alpha)$ on the torus $\beta = \beta_1$ its external expression is

$$20)_a\quad U^{o,m}(\alpha,\beta) = \frac{1}{\pi}\sum_{n=0}^{\infty} \epsilon_n \frac{Q^{n}_{m-1/2}(\coth\beta)}{Q^{n}_{m-1/2}(\coth\beta_1)} \int_{-\pi}^{\pi} f(\alpha_1)\cos n(\alpha-\alpha_1)\, d\alpha_1 \quad \text{where} \begin{pmatrix} 0 \le \beta < \beta_1 \\ -\pi \le \alpha \le \pi \end{pmatrix}$$

The internal potential is

$$20)_b\quad U^{i,m}(\alpha,\beta) = \frac{1}{\pi}\sum_{n=0}^{\infty} \epsilon_n \frac{P^{n}_{m-1/2}(\coth\beta)}{P^{n}_{m-1/2}(\coth\beta_1)} \int_{-\pi}^{\pi} f(\alpha_1)\cos n(\alpha-\alpha_1)\, d\alpha_1 \quad \text{where} \begin{pmatrix} \beta_1 \le \beta \le \infty \\ -\pi \le \alpha \le \pi \end{pmatrix}$$

These potentials, like the series (17) and the integrals (18), (18') and (19), may be taken as referring either to the z-plane of fig 1 or the z'-plane of fig 1'.

(b) Potential of a simple distribution at a cut, $\alpha = \pm\pi$.

The two sides of the cut, $\alpha = \pm\pi$, are generators of both sides of a circular disc by fig 1, but in fig 1' the disc is bent into part of a spherical surface. Assume that the (reduced) density $\bar{\sigma}(\beta)$ at this cut has a finite total charge so that

21)$_a$ $\int_0^\infty \frac{\bar{\sigma}(\beta)}{h(\pi,\beta)}\,d\beta = c\int_0^\infty \frac{\bar{\sigma}(\beta)\,d\beta}{\cosh\beta + 1}$ converges. Assume also that $\bar{\sigma}(0)$ is finite and that if $\bar{\sigma}(\beta)$ becomes infinite at the edge ($\beta = +\infty$) it must be like

21)$_b$
$$\begin{cases} \bar{\sigma}(\beta) \sim C e^{(1-\delta)\beta} \text{ as } \beta \to \infty \text{ where } \delta > 0 \text{ so that} \\ \frac{\bar{\sigma}(\beta)}{h(\pi,\beta)} \sim 2C e^{-\delta\beta} \text{ as } \beta \to \infty. \end{cases}$$

Its potential integral,

22) $$U^m(\alpha,\beta) = 2\int_0^\infty \frac{\bar{\sigma}(\beta_1)}{h(\pi,\beta_1)}\, Q_{m-\frac{1}{2}}(g(\alpha,\beta;\pi,\beta_1))\,d\beta_1,$$ will then converge

when α,β represents any point, for by (15)

$$g(\alpha,\beta;\pi,\beta_1) \equiv \frac{\cosh\beta\cosh\beta_1 + \cos\alpha}{\sinh\beta\,\sinh\beta_1} \to \coth\beta \text{ when } \beta_1 \to \infty,$$ so

the potential integral converges, since (21)$_a$ converges.

If in the integral (22) we introduce for $Q^{(\beta)}_{m-1/2}$ its integral representation (19), the order of integration in

the resulting double integral may be inverted if the μ-path is suitably chosen. Eq (22) then leads to the following representation of the potential U^m of the simple distribution $\bar{\sigma}$ on the suit

$$23)\quad U^m(\alpha,\beta) = \frac{2}{\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \frac{\cos\mu\alpha\,\Gamma(\frac{1}{2}+m-\mu)}{\cos^2\mu\pi\,\Gamma(\frac{1}{2}+m+\mu)}\, Q^{\mu}_{m-1/2}(\coth\beta)\,d\mu \int_0^{\infty} \frac{\bar{\sigma}(\beta_1)}{h(\pi,\beta_1)}\, Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\beta_1$$

To find for what values of μ_1, in the interval $-m-\frac{1}{2}<\mu_1<m+\frac{1}{2}$ this interchange of order of integration is permissible we must find the values of μ_1 for which the β_1-integral converges. Now by Whipple's transformation VI (62)d

$$24)\quad Q^{\mu}_{m-1/2}(\coth\beta) = \cos\mu\pi\sqrt{\frac{\pi\sinh\beta}{2}}\;\Gamma(\tfrac{1}{2}-m+\mu)\,P^{m}_{\mu-1/2}(\cosh\beta)$$

and by VI (65)

$$25)\quad P^{m}_{\mu-1/2}(\cosh\beta) = (-1)^{m+1}\frac{\cot\mu\pi}{\sqrt{\pi}}\,e^{-\beta/2}(1-e^{-2\beta})^{m}\left\{\frac{e^{-\mu\beta}\,\Gamma(\frac{1}{2}+m+\mu)}{\Gamma(1+\mu)}\,F(\tfrac{1}{2}+m,\tfrac{1}{2}+m+\mu,1+\mu;e^{-2\beta})\right.$$

Also by VI (45)

$$\left. -\frac{e^{\mu\beta}\,\Gamma(\frac{1}{2}+m-\mu)}{\Gamma(1-\mu)}\,F(\tfrac{1}{2}+m,\tfrac{1}{2}+m-\mu,1-\mu;e^{-2\beta})\right\}$$

$$26)\quad P^{m}_{\mu-1/2}(\cosh\beta) = \tanh^{m}\tfrac{\beta}{2}\,\cosh^{2\mu-1}\tfrac{\beta}{2}\,\frac{\Gamma(\frac{1}{2}+m+\mu)}{m!\,\Gamma(\frac{1}{2}-m+\mu)}\,F(\tfrac{1}{2}-\mu,\tfrac{1}{2}-\mu+m,m+1;\tanh^2\tfrac{\beta}{2})$$

The latter shows that the convergence of the β_1 integral in (23) is

secured as far as the lower limit, $\beta_1 = 0$, is concerned since by hypothesis $\bar{\sigma}(0)$ is finite.

At the upper limit $\beta_1 \to \infty$, where by (21)$_e$, $\bar{\sigma}/h \sim Ce^{-\delta\beta}$, we find by (24) and (25) the convergence requires μ_1 to satisfy the two inequalities

27) $\quad -\delta < \mu_1 < \delta \quad$ and $\quad -(m+\frac{1}{2}) < \mu_1 < m+\frac{1}{2}$

These could always be satisfied by taking the integral up the imaginary axis of μ ($\mu_1 = 0$).

An integral representation of a function $f(\beta)$ for the range $0 < \beta < \infty$ VIII (36)$_e$ is

28) $$f(\beta) = \frac{-1}{\pi^2 i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu \sin\mu\pi\, \Gamma(\frac{1}{2}+m-\mu)}{\cos^2\mu\pi\, \Gamma(\frac{1}{2}+m+\mu)} Q^{\mu}_{m-1/2}(\coth\beta)\, d\mu \int_0^\infty f(\beta_1)\, Q^{\mu}_{m-1/2}(\coth\beta_1)\, d\beta_1$$

where $f(\beta)$ satisfies the following conditions

29)$_a$ $\int_0^\infty |f(\beta)|\, d\beta$ converges

29)$_b$ $f(\beta) \sim Ce^{-\delta\beta}$ as $\beta \to \infty$ and the path in (28) is any line in the plane of the complex variable $\mu \equiv \mu_1 + i\mu_2$, which is parallel to the imaginary axis and lies in the strip determined by the two inequalities

29)$_c$ $\quad -\delta < \mu_1 < \delta \quad$ and $\quad -(m+1/2) < \mu_1 < m+1/2$.

The imaginary axis is always a permissible path.

Comparing 29)$_c$ with (21)$_e$ shows that the function $\dfrac{\bar{\sigma}(\beta)}{h(\pi,\beta)}$ is

of the class $f(\beta)$ which may be represented as in (28). Also as shown by (27) the integral (23) appears as an integral representation of type (28) of the function $\overset{m}{U}(\alpha,\beta)$ considered as a function of β.

If $F(\beta)$ denotes the potential at the cut, then by (23)

$$30)\quad \overset{m}{U}(\pm\pi,\beta) \equiv F(\beta) = \frac{2}{\pi i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \frac{\Gamma(\frac{1}{2}+m-\mu)}{\cos\mu\pi\,\Gamma(\frac{1}{2}+m+\mu)} Q^{\mu}_{m-1/2}(\coth\beta)\,d\mu \int_0^\infty \frac{\bar{\sigma}(\beta_1)}{h(\pi,\beta_1)} Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\beta_1$$

Since the representation of type (28) is unique, the eq (30) shows that

$$31)\quad \int_0^\infty \frac{\bar{\sigma}(\beta_1)}{h(\pi,\beta_1)} Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\beta_1 = -\frac{\mu\tan\mu\pi}{2\pi}\int_0^\infty F(\beta_1)\, Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\beta_1$$

By use of this relation the potential (23) which is expressed in terms of the charge density at the cut, gives the following expression in terms of the potential values at the cut

$$32)_a\quad \overset{m}{U}(\alpha,\beta) = \frac{-1}{\pi^2 i}\int_{\mu_1-i\infty}^{\mu_1+i\infty} \frac{\mu\sin\mu\pi\cos\mu\alpha\,\Gamma(\frac{1}{2}-m-\mu)}{\cos^3\mu\pi\;\Gamma(\frac{1}{2}+m+\mu)} Q^{\mu}_{m-1/2}(\coth\beta)\,d\mu \int_0^\infty F(\beta_1)\, Q^{\mu}_{m-1/2}(\coth\beta_1)\,d\beta_1$$

or by VI (62)d

$$32)_b\quad \overset{m}{U}(\alpha,\beta) = \frac{(-1)^{m+1}\sqrt{\sinh\beta}}{2i}\int \frac{\mu\sin\mu\pi\cos\mu\alpha\,\Gamma(\frac{1}{2}-m+\mu)}{\cos^2\mu\pi\;\Gamma(\frac{1}{2}+m+\mu)} P^{-m}_{\mu-\frac{1}{2}}(\cosh\beta)\;\mu\int_0^\infty F(\beta_1)\sqrt{\sinh\beta_1}\, P^{-m}_{\mu-1/2}(\cosh\beta_1)\,d\beta_1$$

where $-m-\frac{1}{2} < \mu_1 < m+\frac{1}{2}$

This makes the normal derivative of the reduced potential vanish on the arc $\widehat{A_0B_0C}$ of fig 1′ ($\alpha = 0$), but the normal derivative of the ordinary potential $V^m = \dfrac{U^m}{\sqrt{\rho}} \cos m(\phi - \phi_m)$ does not vanish there, although it does vanish with the interpretation of fig 1 when $\alpha = 0$, ($x = 0$ and $\rho > c$.).

The inversion of the equation (30), giving the density in terms of the potential values at the cut is obtained from the relation between the transforms of the two functions $F(\beta)$ and $\dfrac{\bar\sigma(\beta)}{h(\pi,\beta)}$ given in (31).

It is

33) $$\frac{\bar\sigma(\beta)}{h(\pi,\beta)} = \frac{1}{2\pi^3 i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu^2 \sin^2 \mu\pi \, \Gamma(\frac{1}{2}+m-\mu)}{\cos^3 \mu\pi \, \Gamma(\frac{1}{2}+m+\mu)} Q^{\mu}_{m-1/2}(\coth\beta)\, d\mu \int_0^\infty F(\beta_1)\, Q^{\mu}_{m-1/2}(\coth\beta_1)\, d\beta_1$$

$$= \frac{(-1)^m \sqrt{\sinh\beta}}{4\pi i}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu^2 \tan^2 \mu\pi \frac{\Gamma(\frac{1}{2}-m+\mu)}{\Gamma(\frac{1}{2}+m+\mu)} P^{m}_{\mu-1/2}(\cosh\beta)\, d\mu \int_0^\infty F(\beta_1)\sqrt{\sinh\beta_1}\, P^{m}_{\mu-1/2}(\cosh\beta_1)\, d\beta_1$$

$-m-\frac{1}{2} < \mu_1 < m+\frac{1}{2}$

4. Prolate Spheroidal Coordinates and their Inversion.

The $x\rho$-half-plane of fig 1 is represented on the semi-infinite strip of the ω-plane, $0<\alpha<\pi$, $0<\beta<\infty$; of fig 2, by the equation

1) $\quad z \equiv x+i\rho = -c\cos\omega = -c\cos(\alpha+i\beta) \qquad (c>0)$

or

1)$_a$ $\quad x = -c\cos\alpha\cosh\beta$

1)$_b$ $\quad \rho = c\sin\alpha\sinh\beta$

$$\left.\begin{array}{l} x=-c\cos\alpha\cosh\beta \\ \rho = c\sin\alpha\sinh\beta \end{array}\right\} \text{ or } \left\{\begin{array}{l} \dfrac{x^2}{c^2\cosh^2\beta}+\dfrac{\rho^2}{c^2\sinh^2\beta}=1 \\ \dfrac{x^2}{c^2\cos^2\alpha}-\dfrac{\rho^2}{c^2\sin^2\alpha}=1 \end{array}\right\}$$

1)$_c$ $\quad \sqrt{dx^2+d\rho^2} = \dfrac{\sqrt{d\alpha^2+d\beta^2}}{h}$ where $h(\alpha,\beta) = \dfrac{1}{c\sqrt{\sinh^2\beta+\sin^2\alpha}}$ and $r \equiv \sqrt{x^2+\rho^2} = c\sqrt{\sinh^2\beta+\cos^2\alpha}$

1)$_d$ $\quad \dfrac{1}{\rho^2 h^2} = \dfrac{1}{\sin^2\alpha}+\dfrac{1}{\sinh^2\beta}$

Euler's equation for the reduced potential U^m

2)$_a$ $\quad \left(D_x^2+D_\rho^2+\dfrac{1/4-m^2}{\rho^2}\right)U^m = 0 \quad$ becomes

2)$_b$ $\quad \left[D_\alpha^2+D_\beta^2+\left(\tfrac{1}{4}-m^2\right)\left(\dfrac{1}{\sin^2\alpha}+\dfrac{1}{\sinh^2\beta}\right)\right]U^m = 0$

This has solutions of the form $U = u(\alpha)\,v(\beta)$ where

3)$_a$ $\quad u''(\alpha)+\left(\dfrac{\frac{1}{4}-m^2}{\sin^2\alpha}+\nu^2\right)u(\alpha) = 0$

3)$_b$ $\quad v''(\beta)+\left(\dfrac{\frac{1}{4}-m^2}{\sinh^2\beta}-\nu^2\right)v(\beta) = 0$

Let

$\xi = \cos\alpha$ and $u(\alpha) = \sqrt{\sin\alpha}\, y(\xi)$ in $(3)_a$

$\xi = \cosh\beta$ and $v(\beta) = \sqrt{\sinh\beta}\, y(\xi)$ in $(3)_b$

The two equations (3) transform into the same equation

4) $$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dy}{d\xi}\right] + \left[\nu - \frac{1}{4} - \frac{m^2}{1-\xi^2}\right] y = 0$$ so that eq $(2)_b$

has as particular solutions

5) $$U^m_{(\alpha,\beta)} = \sqrt{\sin\alpha \sinh\beta}\left[A\, T^m_\nu(\cos\alpha) + B\, T^m_\nu(-\cos\alpha)\right]\left[C\, P^m_\nu(\cosh\beta) + D\, Q^m_\nu(\cosh\beta)\right]$$

The condition for no sources on the x axis becomes

$6)_a$ $U^m \to 0$, like $\beta^{m+\frac{1}{2}}$ when $\beta \to 0$ when $x^2 < c^2$ IX $(13)_b$

$6)_b$ $U^m \to 0$ like $\sin\alpha^{m\frac{1}{2}}$ when $\alpha \to 0$ or π - - $x^2 > c^2$

The condition for no sources at infinity becomes

7) $U^m \to 0$ like $e^{-(m+\frac{1}{2})\beta}$ when $\beta \to \infty$ IX $(13)_a$

The potentials to be obtained may also be interpreted on the z'-half-plane of fig 1' or fig 1'' where $z' = x' + i\rho'$. This is represented on the same w-strip by the inversion

8) $$(z - x_0)z' - c^2 \quad \text{or} \quad z' = \frac{c}{\cos w + \frac{x_0}{c}}$$

The curves sketched in fig 1' and 1 are inversions of ellipses of fig 1, the locus β = constant. These have the polar equation

$8)_a$ $$r' - \frac{c}{\sinh^2\beta - \sinh^2\beta_0}\left[-\cosh\beta_0 \cos\theta' \pm \sqrt{1 - \frac{\sinh^2\beta_0}{\sinh^2\beta}\sin\theta'}\right]$$

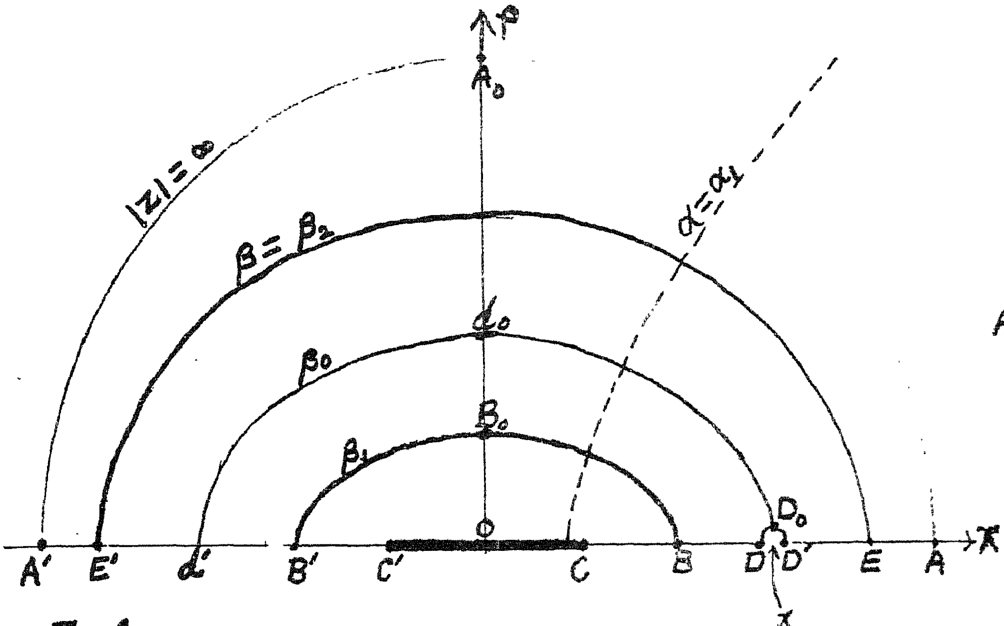


Fig 1

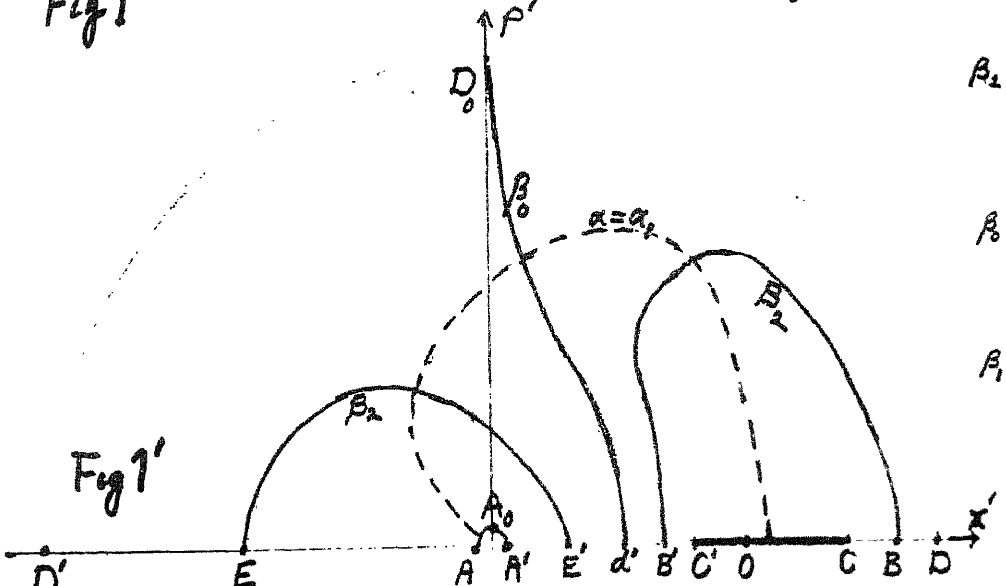


Fig 1'

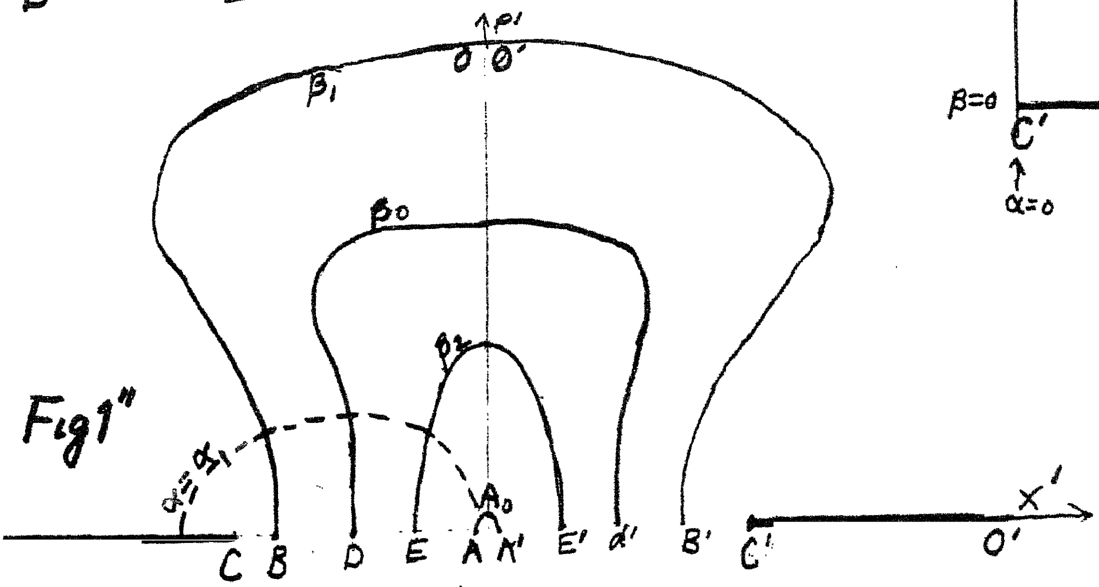


Fig 1''

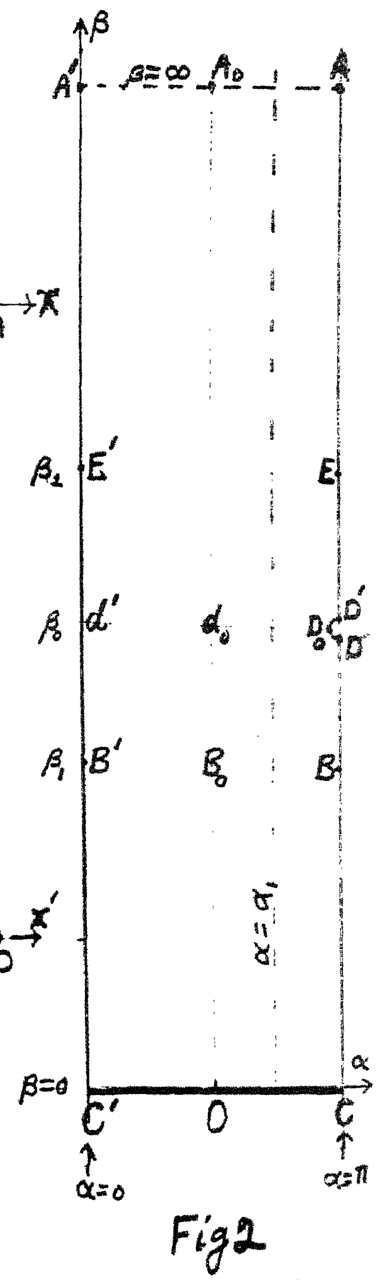


Fig 2

ω-strip

Prolate spheroidal coordinates and their inversions.

where $z' = x' + i\rho' = r' e^{i\theta'}$ and $x_0 = c\cosh\beta_0$.

The open curve $\beta = \beta_0$ in fig 1' has the equation

$$8)_b \quad r' = \frac{c}{2\sinh^2\beta_0}\left[\frac{\cosh\beta_0}{\cosh\theta'} - \frac{\cos\theta'}{\cosh\beta_0}\right]$$

The curves sketched in fig 1'' are for the case $x_0 = 2$. The one dotted curve in all figures is the locus $\alpha = \alpha_1$, the hyperbola of fig 1 or its inversions.

(a) Potential given on a Prolate Spheroid or its Inversion

The same argument applies here as in polar coordinates. Since m is a given non-negative integer the only values of ν which permit a potential of the form (5) to have no sources on the x axis where $x > c$ $(\alpha = \pi)$ and also when $x < -c$ $(\alpha = 0)$ are $\nu = n \geq m$ where n is integral. The corresponding solution of $(3)_a$ is $\sqrt{\sin\alpha}\, T_n^m(\cos\alpha)$

Also by VI (43)

$$9)_a \quad Q_n^m(\cosh\beta) = \frac{(-1)^m\sqrt{\pi}\,\sinh^m\beta\,(n+m)!}{2^{n+1}(\cosh\beta+1)^{n+m+1}\,\Gamma(n+\frac{3}{2})} F\left(n+1, n+m+1, 2n+2; \frac{2}{\cosh\beta+1}\right)$$

and by VI $(32)_b$

$$9)_b \quad P_n^m(\cosh\beta) = \frac{(-1)^{n-m}\sinh^m\beta\,(n+m)!}{2^m\, m!\,(n-m)!} F\left(m-n, n+m+1; m+1; \frac{\cosh\beta+1}{2}\right)$$

this hypergeometric function being a polynomial.

Also by VI (7)

$$9)_c \quad \sqrt{\sinh\beta}\left\{ Q_n^m(\cosh\beta)\, D_\beta\left[\sqrt{\sinh\beta}\, P_n^m(\cosh\beta)\right] - P_n^m(\cosh\beta)\, D_\beta\left[\sqrt{\sinh\beta}\, Q_n^m(\cosh\beta)\right]\right\} =$$

$$= \frac{(-1)^m (n+m)!}{(n-m)!}$$

Hence as in the case of polar coordinates we define

$$10)_a \begin{cases} \mathcal{U}_n^m(\alpha) = C_n^m \sqrt{\sin\alpha}\, T_n^m(\cos\alpha) \quad \text{where } C_n^m = \sqrt{(n+\tfrac{1}{2})\dfrac{(n-m)!}{(n+m)!}} \\ \text{so that} \\ \int_0^\pi \mathcal{U}_{n_1}^m(\alpha)\, \mathcal{U}_{n_2}^m(\alpha)\, d\alpha = \delta_{n_1 n_2} \end{cases}$$

and

$$10)_b \quad \mathcal{V}_n^{om}(\beta) = \sqrt{\sinh\beta}\, Q_n^m(\cosh\beta) \qquad \text{and} \qquad \mathcal{V}_n^{im}(\beta) = \sqrt{\sinh\beta}\, P_n^m(\cosh\beta)$$

Then $(9)_c$ may be written

$$11) \quad \gamma_n^m \equiv \mathcal{V}_n^{om}(\beta)\, \mathcal{V}_n^{\prime im}(\beta) - \mathcal{V}_n^{\prime om}(\beta)\, \mathcal{V}_n^{im}(\beta) = (-1)^m \frac{(n+m)!}{(n-m)!}$$

Since the integer n is $\geq m$, the F-function in $(9)_b$ being a polynomial, $\mathcal{V}^{im}(\beta)$ will vanish with β like $\beta^{m+\frac{1}{2}}$ thus satisfying the requirement $(6)_a$ that there is no source on the focal line $C'OC$ where $\kappa^2 < c^2$ $(\beta = 0)$. The internal harmonics with respect to the spheroid $\beta = \beta_1$ must therefore be of the form

$$12)_a \quad U_n^{im}(\alpha,\beta) = \mathcal{V}_n^{im}(\beta)\, \mathcal{V}_n^{om}(\beta_1)\, \mathcal{U}_n^m(\alpha) = C_n^m \sqrt{\sin\alpha \sinh\beta \sinh\beta_1}\, P_n^m(\cosh\beta)\, Q_n^m(\cosh\beta_1)\, T_n^m(\cos\alpha)$$

where $0 \leq \beta \leq \beta_1$

The external harmonics must be of the form

for $\beta_1 < \beta \leq \infty$

$$12)_e \quad \overset{o\,m}{\underset{n}{U}}(\ ,\beta) - \overset{i\,m}{\underset{n}{V}}(\beta_1)\overset{o\,m}{\underset{n}{V}}(\beta)\overset{m}{\underset{n}{\mathcal{U}}}(\alpha) = \overset{m}{\underset{n}{C}}\sqrt{\sin\alpha\sinh\beta\sinh\beta_1}\,\overset{m}{\underset{n}{P}}(\cosh\beta_1)\overset{m}{\underset{n}{Q}}(\cosh\beta)\overset{m}{\underset{n}{T}}(\cos\alpha)$$

Eq(9)$_e$ shows that this satisfies the condition (4) that there be no sources at infinity, while the condition (6)$_e$ that there be no sources on that part of the x axis where $x^2 > c^2$ is satisfied for U^o ad U^i by the character of the factor $\sqrt{\sin\alpha}\overset{m}{\underset{n}{T}}(\cos\alpha)$ This vanishes with $\sin\alpha$ like $\sin\alpha^{m+\frac{1}{2}}$ as shown by

$$\sqrt{\sin\alpha}\,\overset{m}{\underset{n}{T}}(\cos\alpha) = \frac{\sin\alpha^{m+\frac{1}{2}}\,(n+m)!}{2^m\, m!\,(n-m)!}\,F\left(m-n,\ n+m+1,\ m+1;\ \frac{1-\cos\alpha}{2}\right)$$

Hence the (reduced) potential (12) is that of a simple distribution on the prolate spheroid whose density is given by

$$13)\quad \frac{4\pi\bar{\sigma}(\alpha}{h(\alpha,\beta_1)} = -\left(D_\beta U^o\right)_{\beta=\beta_1+0} + \left(D_\beta U^i\right)_{\beta=\beta_1-0} \quad \text{or by (11)}$$

$$14)\quad \frac{4\pi\bar{\sigma}(\alpha)}{h(\alpha,\beta_1)} = \overset{m}{\underset{n}{\gamma}}\ \overset{m}{\underset{n}{\mathcal{U}}}(\alpha) = (-1)^m\frac{(n+m)!}{(n-m)!}\cdot\overset{m}{\underset{n}{\mathcal{U}}}(\alpha)$$

The two forms of (12) must be equivalent to

$$15)\quad \overset{m}{V}(\alpha,\beta) = 2\int_0^\pi \frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1,\beta_1)}\,Q_{m-\frac{1}{2}}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right)d\alpha_1$$

where

$$16)\quad g(\alpha,\beta;\alpha_1,\beta_1) =$$

$$= \frac{\cos^2\alpha + \sinh^2\beta + \cos^2\alpha_1 + \sinh^2\beta_1 - 2\cos\alpha\cosh\beta\cos\alpha_1\cosh\beta_1}{2\sin\alpha\sinh\beta\sin\alpha_1\sinh\beta_1}$$

Using in (15) the value of $\bar{\sigma}/h$ from (14) and equating U to U^i or U^o

shows that $\mathcal{U}_n^m(\alpha)$ is a solution of the integral equation

$$17)_a \quad \int_0^\pi \mathcal{U}_n^m(\alpha_1)\, Q_{m-1/2}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right)d\alpha_1 = \frac{2\pi}{\gamma_n^m}\, \mathcal{V}^{im}(\beta)\,\mathcal{V}^{om}(\beta_1)\,\mathcal{U}_n^m(\alpha) \quad \text{if } 0 \leq \beta \leq \beta_1$$

that is

$$= \frac{2\pi}{\gamma_n^m}\, \mathcal{V}^{im}(\beta_1)\,\mathcal{V}^{om}(\beta)\,\mathcal{U}_n^m(\alpha) \quad \text{if } \beta_1 \leq \beta < \infty$$

$$17)_b \quad \int_0^\pi \sqrt{\sin\alpha_1}\, T_n^m(\cos\alpha_1)\, Q_{m-1/2}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right)d\alpha_1 =$$

$$= 2\pi\sqrt{\sinh\beta\sinh\beta_1}\,.(-1)^m \frac{(n-m)!}{(n+m)!}\, P_n^m(\cosh\beta)\, Q_n^m(\cosh\beta_1)\, \sqrt{\sin\alpha}\, T_n^m(\cos\alpha) \quad \text{if } 0 \leq \beta \leq \beta_1$$

This integral is the Fourier coefficient of the series of normal functions $\mathcal{U}_n^m(\alpha)$ which represents $Q_{m-\frac{1}{2}}(g)$ as a function of α. Hence

$$18) \quad Q_{m-1/2}\left(g(\alpha,\beta;\alpha_1,\beta_1)\right) =$$

$$= (-1)^m 2\pi\sqrt{\sin\alpha\sin\alpha_1\sinh\beta\sinh\beta_1}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\left[\frac{(n-m)!}{(n+m)!}\right]^2 T_n^m(\cos\alpha)\, T_n^m(\cos\alpha_1)\, P_n^m(\cosh\beta)\, Q_n^m(\cosh\beta_1)$$

when $0 \leq \beta \leq \beta_1$

The (reduced) potential which has assigned values, $f(\alpha)$, on the prolate ellipsoid $\beta = \beta_1$ or any of its inversions is given by

19) $$U^{o\,m}_{(\alpha,\beta)} = \sqrt{\frac{\sin\alpha\,\sinh\beta}{\sinh\beta_1}}\sum_{n=m}^{\infty}\left(n+\tfrac{1}{2}\right)\frac{(n-m)!}{(n+m)!}\,\frac{Q^m_n(\cosh\beta)}{Q^m_n(\cosh\beta_1)}\,T^m_n(\cos\alpha)\int_0^{\pi} f(\alpha_1)\sqrt{\sin\alpha_1}\,T^m_n(\cos\alpha_1)\,d\alpha_1$$

when $\beta_1 \leq \beta \leq \infty$, that is outside the ellipsoid. For internal points where $c \leq \beta \leq \beta_1$ the Q-functions are replaced by P-functions.

(b) Potential given on one Sheet of a Two-sheeted Hyperboloid or on any of its Inversions.

This is the locus $\alpha = \alpha_1$ $(0 < \beta < \infty)$ which is shown as a dotted curve in all the figures 1, 2, 1', ad 1''. The region $\alpha_1 < \alpha < \pi$ will be called the internal region, which in fig 1 is bounded by a hyperbolic arc and part of the positive x-axis. In fig 1'' this is also properly called internal but it is a misnomer for fig 1'.

For this problem it is convenient to use equations (65) of section VI (for $0 < \beta < \infty$)

20)$_a$ $$Q^m_\nu(\cosh\beta) = (-1)^m\sqrt{\pi}\,(1-e^{-2\beta})^m\,e^{-(\nu+1)\beta}\,\frac{\Gamma(\nu+m+1)}{\Gamma(\nu+\frac{3}{2})}\,F\!\left(m+\tfrac{1}{2},\,\nu+m+1,\,\nu+\tfrac{3}{2};\,e^{-2\beta}\right)$$

20)$_b$ $$P^m_\nu(\cosh\beta) = \frac{(-1)^m}{\sqrt{\pi}}(1-e^{-2\beta})^m\tan\nu\pi\left\{e^{-(\nu+1)\beta}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu+\frac{3}{2})}F\!\left(m+\tfrac{1}{2},\,\nu+m+1,\,\nu+\tfrac{3}{2};\,e^{-2\beta}\right)\right.$$
$$\left. - e^{\nu\beta}\frac{\Gamma(m-\nu)}{\Gamma(\frac{1}{2}-\nu)}F\!\left(m+\tfrac{1}{2},\,m-\nu,\,\tfrac{1}{2}-\nu;\,e^{-2\beta}\right)\right\}$$

This equation is equivalent to the fundamental relation

$$21)\quad \underset{\nu}{P}^{m}(\cosh\beta) = \frac{\tan\nu\pi}{\pi}\left[\underset{\nu}{Q}^{m}(\cosh\beta) - \underset{-\nu-1}{Q}^{m}(\cosh\beta)\right] \qquad \text{VI } (6)_b$$

We also require VI (45)

$$22)_a\quad \underset{\nu}{P}^{m}(\cosh\beta) = \frac{\tanh^{m}\beta_{/2}\cosh^{2\nu}\beta_{/2}}{m!}\frac{\Gamma(\nu+m+1)}{\Gamma(\nu-m+1)}F(-\nu,\, m-\nu,\, m+1;\, \tanh^2\beta_{/2})$$

By VI (46)′

$$22)_b\quad \underset{\nu}{Q}^{m}(\cosh\beta) = \underset{\nu}{P}^{m}(\cosh\beta)\left[\tfrac{1}{2}\log\coth\tfrac{\beta}{2} + \pi\cot\nu\pi\right]$$

$$+\tfrac{1}{2}\Gamma(\nu+1)\Gamma(\nu+m+1)\tanh^{m}\beta_{/2}\cosh^{2\nu}\beta_{/2}\sum_{s=1}^{s=m>0}\frac{(-1)^{s}\coth^{2s}\beta_{/2}\,\Gamma(s)}{\Gamma(s+\nu+1)\Gamma(s+\nu-m+1)\Gamma(1+m-s)}$$

+ a series of positive powers of $\tanh^2\beta_{/2}$.

Also by VI $(12)_a$ for $0 \leq \alpha < \pi$

$$23)_a\quad \underset{\nu}{T}^{m}(\cos\alpha) = \frac{\sin^{m}\alpha\,\Gamma(\nu+m+1)}{2^{m}\,m!\,\Gamma(\nu-m+1)}F\left(m-\nu,\, \nu+m+1,\, m+1;\, \frac{1-\cos\alpha}{2}\right)$$

which gives $\underset{\nu}{T}^{m}(-\cos\alpha)$ on replacing α by $\pi-\alpha$.

Eq VI $(26)_i$ is equivalent to

$$23)_b\quad \sqrt{\sin\alpha}\left\{\underset{\nu}{T}^{m}(-\cos\alpha)\,D_{\alpha}\left(\sqrt{\sin\alpha}\,\underset{\nu}{T}^{m}(\cos\alpha)\right) - \underset{\nu}{T}^{m}(\cos\alpha)\,D_{\alpha}\left(\sqrt{\sin\alpha}\,\underset{\nu}{T}^{m}(-\cos\alpha)\right)\right\} = \frac{2}{\Gamma(\nu-m+1)\Gamma(-\nu-m)}$$

A simple distribution with (reduced) density $\bar{\sigma}(\beta)$ on the hyperboloid (or locus $\alpha=\alpha_1$) gives rise to a (reduced) potential $\overset{m}{V}(\alpha,\beta)$ where, if $\lim\limits_{\beta\to\infty}\dfrac{\bar{\sigma}(\beta)}{h(\alpha_1,\beta)}e^{-(m+\frac{1}{2})\beta} = 0$,

24) $$U^{m}(\alpha,\beta) = 2\int_0^\infty \frac{\bar{\sigma}(\beta)}{h(\alpha_1,\beta)} Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta))\,d\beta,$$ where g is given by (16).

and

25) $$\frac{4\pi\bar{\sigma}(\beta)}{h(\alpha_1,\beta)} = -(D_\alpha U^m)_{\alpha=\alpha_1+0} + (D_\alpha U^m)_{\alpha=\alpha_1-0}$$

Consider the continuous normal solutions for $0 \le \beta \le \infty$

26)$_a$ $$U_\nu^{om}(\alpha,\beta) = \sqrt{\sin\alpha\,\sin\alpha_1}\, T_\nu^m(-\cos\alpha_1)\, T_\nu^m(\cos\alpha)\sqrt{\sinh\beta}\, P_\nu^m(\cosh\beta)$$ where $0 \le \alpha \le \alpha_1$

and

26)$_b$ $$U_\nu^{1,m}(\alpha,\beta) = \sqrt{\sin\alpha\,\sin\alpha_1}\, T_\nu^m(\cos\alpha_1)\, T_\nu^m(-\cos\alpha)\cdot\sqrt{\sinh\beta}\, P_\nu^m(\cosh\beta)$$ where $\alpha_1 \le \alpha \le \pi$

This is the potential of a density on the locus $\alpha=\alpha_1$ where by (23)$_b$ and (25)

26)$_c$ $$\frac{4\pi\sigma_\nu(\beta)}{h(\alpha_1,\beta)} = \frac{2}{\Gamma(\nu-m+1)\Gamma(-\nu-m)}\sqrt{\sinh\beta}\, P_\nu^m(\cosh\beta).$$

If $P_\nu^m(\cosh\beta)$ were replaced by $Q_\nu^m(\cosh\beta)$, the corresponding potential would also have sources on the focal line ($\beta=0$) as shown by (22)$_b$. The potential (26) has no sources on the focal line for it satisfies the condition (6)$_a$ as shown by (22)$_a$. Also eq (6)$_b$ for no sources on that part of the x axis where $|x|>c$ is taken care of by the T-function as shown by (23)$_a$. Hence (26)$_{a,c}$ represents the potential of the simple distribution, (26)$_c$, on the locus $\alpha=\alpha_1$, regardless of the value of the parameter ν. This is limited to a certain complex domain by the requirement that the potential integral (24) be convergent. When $\beta\to\infty$ $\sqrt{\sinh\beta}\,P_\nu^m(\cosh\beta) \to A e^{(\nu+\frac{1}{2})\beta} + B e^{-(\nu+\frac{1}{2})\beta}$

Hence σ/h becomes infinite like $e^{(\nu+\frac{1}{2})\beta}$ or $e^{-(\nu+\frac{1}{2})\beta}$ when $\beta\to\infty$ but as shown in IX (15) its potential integral (24) will converge for every finite point (α,β) if $\nu(\equiv\nu_1+i\nu_2)$ is represented by a point in the strip of the ν-plane

27) $\quad -m-1<\nu_1<m \quad$ and $\quad -\infty<\nu_2<\infty$.

Using the density $(26)_c$ in (24) shows that $P_\nu^m(\cosh\beta)$ is a solution of the homogeneous integral equation

28) $$\int_0^\infty \sqrt{\sinh\beta_1}\,P_\nu^m(\cosh\beta_1)\,Q_{m-1/2}(q(\alpha,\beta;\alpha_1\beta_1))\,d\beta_1=$$

$$=\pi\sqrt{\sin\alpha\sin\alpha_1}\,\Gamma(\nu-m+1)\Gamma(-\nu-m)\,T_\nu^m(-\cos\alpha_1)\,T_\nu^m(\cos\alpha)\,.\sqrt{\sinh\beta}\,P_\nu^m(\cosh\beta)$$

when $0<\alpha<\alpha_1$

In section VIII integral representations, eq $(40)_{a,b,c}$, were obtained for a function $f(\beta)$ for $0<\beta<\infty$.

Equ $(40)_c$ is

29) $$f(\beta)=\frac{(-1)^m\sqrt{\sinh\beta}}{2i}\int_{\nu_1-i\infty}^{\nu_1+i\infty}(\nu+\tfrac{1}{2})\cot\nu\pi\,\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\,P_\nu^m(\cosh\beta)\,d\nu\int_0^\infty f(\beta')\sqrt{\sinh\beta'}\,P_\nu^m(\cosh\beta')\,d\beta'$$

where ν_1 is given by (27). Hence considering $Q_{m-\frac{1}{2}}(q)$ as a function of β its transform is given by (28) so that when $0\leq\alpha\leq\alpha_1$

30) $$Q_{m-1/2}(q(\alpha,\beta;\alpha_1,\beta_1))=$$

$$=-\pi^2\frac{\sqrt{\sin\alpha\sinh\beta\sin\alpha_1\sinh\beta_1}}{2i}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\frac{(\nu+\frac{1}{2})\cos\nu\pi\,\Gamma^2(\nu-m+1)}{\sin^2\nu\pi\,\Gamma^2(\nu+m+1)}\,T_\nu^m(\cos\alpha)\,T_\nu^m(-\cos\alpha_1)\,P_\nu^m(\cosh\beta)\,P_\nu^m(\cosh\beta_1)\,d\nu$$

$-m-1<\nu_1<m$

Using this, the potential integral (24) becomes

31) $U^m_{(\alpha,\beta)} =$

$$= -\frac{\pi^2}{i}\sqrt{\sin\alpha\sin\alpha_1\sinh\beta}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\frac{(\nu+\frac{1}{2})\cos\nu\pi\,\Gamma^2(\nu-m+1)}{\sin^2\nu\pi\,\Gamma^2(\nu+m+1)}T^m_\nu(\cos\alpha)\,T^m_\nu(-\cos\alpha_1)\,P^m_\nu(\cosh\beta)\,d\nu\,\cdot$$

$-m-1<\nu_1<m$

$$\cdot\int_0^\infty\frac{\bar{\sigma}(\beta_1)\sqrt{\sinh\beta_1}}{h(\alpha_1,\beta_1)}P^m_\nu(\cosh\beta_1)\,d\beta_1$$

provided $0\leq\alpha<\alpha_1$, these two being interchanged otherwise. If the assigned values of U^m at $\alpha=\alpha_1$ is $F(\beta)$, this gives

32) $U^m(\alpha_1,\beta) = F(\beta) =$

$$= -\frac{\pi^2\sin\alpha_1}{i}\sqrt{\sinh\beta}\int_{\nu_1-i\infty}^{\nu_1+i\infty}(\nu+\tfrac{1}{2})\frac{\cos\nu\pi\,\Gamma^2(\nu-m+1)}{\sin^2\nu\pi\,\Gamma^2(\nu+m+1)}T^m_\nu(\cos\alpha_1)\,T^m_\nu(-\cos\alpha_1)\,P^m_\nu(\cosh\beta)\,d\nu\,\cdot$$

$$\cdot\int_0^\infty\frac{\bar{\sigma}(\beta_1)\sqrt{\sinh\beta_1}}{h(\alpha_1,\beta_1)}P^m_\nu(\cosh\beta_1)\,d\beta_1$$

which is a case of the integral representation (29), which is unique, so that, comparing (29) and (32) gives

33) $$\frac{\pi^2}{\sin\nu\pi}\frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)}\cdot T^m_\nu(\cos\alpha_1)\,T^m_\nu(-\cos\alpha_1)\int_0^\infty\frac{\bar{\sigma}(\beta_1)}{h(\alpha_1\beta_1)}\sqrt{\sin\alpha_1}\,P^m_\nu(\cosh\beta_1)\,d\beta_1 =$$

$$= \frac{(-1)^m}{2\sin\alpha_1}\int_0^\infty F(\beta)\sqrt{\sinh\beta_1}\,P^m_\nu(\cosh\beta_1)\,d\beta_1$$

Using this in (31) gives

34) $U^m(\alpha,\beta) =$

$$= \frac{(-1)^m \sqrt{\sinh\beta}}{2i} \sqrt{\frac{\sin\alpha}{\sin\alpha_1}} \int_{\nu_1-i\infty}^{\nu_1+i\infty} (\nu+\tfrac{1}{2}) \cot\nu\pi \, \frac{\Gamma(\nu-m+1)}{\Gamma(\nu+m+1)} \cdot \frac{T_\nu^m(\cos\alpha)}{T_\nu^m(\cos\alpha_1)} P_\nu^m(\cosh\beta)\, d\nu \int_0^\infty F(\beta_1)\sqrt{\sinh\beta_1}\, P_\nu^m(\cosh\beta_1)\, d\beta_1$$

$-m+1 < \nu_1 < m$ which gives the reduced potential for $0 \leq \alpha \leq \alpha_1$ which reduces to $F(\beta)$ on the hyperboloid (or locus $\alpha = \alpha_1$).

The solution for the remaining region $\alpha_1 \leq \alpha \leq \pi$ is obtained on replacing $\frac{T_\nu^m(\cos\alpha)}{T_\nu^m(\cos\alpha_1)}$ by $\frac{T_\nu^m(-\cos\alpha)}{T_\nu^m(-\cos\alpha_1)}$

5 Oblate Spheroidal Coordinates and their Inversion

The $x\rho$-half-plane of fig 1 is represented upon the semi-infinite strip, $0<\alpha<\pi$, $0<\beta<\infty$ of the ω-plane of fig 2 by the equation

1) $z \equiv x + i\rho = ic\sin\omega = ic\sin(\alpha+i\beta)$ where $c>0$

or

$$\left.\begin{aligned} 1)_a\quad & x = -c\cos\alpha\sinh\beta \\ 1)_b\quad & \rho = c\sin\alpha\cosh\beta \end{aligned}\right\} \text{ or } \left\{\begin{aligned} & \frac{x^2}{c^2\sinh^2\beta} + \frac{\rho^2}{c^2\cosh^2\beta} = 1 \\ & \frac{-x^2}{c^2\cos^2\alpha} + \frac{\rho^2}{c^2\sin^2\alpha} = 1 \end{aligned}\right\}$$

1)$_c$ $\sqrt{dx^2+d\rho^2} = \dfrac{\sqrt{d\alpha^2+d\beta^2}}{h}$ where $h(\alpha,\beta) = \dfrac{1}{c\sqrt{\cosh^2\beta - \sin^2\alpha}}$

so that

1)$_d$ $$\frac{1}{\rho^2 h^2} = \frac{1}{\sin^2\alpha} - \frac{1}{\sinh^2\beta}$$

Euler's equation for the reduced potential,

2)$_a$ $\left(\mathfrak{D}_x^2 + \mathfrak{D}_\rho^2 + \dfrac{1/4 - m^2}{\rho^2}\right)U^m = 0$ becomes

2)$_b$ $\left[\mathfrak{D}_\alpha^2 + \mathfrak{D}_\beta^2 + (1/4 - m^2)\left(\dfrac{1}{\sin^2\alpha} - \dfrac{1}{\cosh^2\beta}\right)\right]U^m = 0$ which has solutions of the form $U = u(\alpha)\,v(\beta)$ where

3)$_a$ $$u''(\alpha) + \left[\frac{1/4 - m^2}{\sin^2\alpha} + \mu^2\right]u(\alpha) = 0$$

3)$_b$ $$v''(\beta) + \left[\frac{m^2 - 1/4}{\cosh^2\beta} - \mu^2\right]v(\beta) = 0$$

To obtain solutions suitable when the potential is given on an oblate spheroid ($\beta = \beta_1$) make the substitutions

$\xi = \cos\alpha$ and $u(\alpha) = \sqrt{\sin\alpha}\; y(\xi)$ in $(3)_a$

$\xi = i\sinh\beta$ and $v(\beta) = \sqrt{\cosh\beta}\; y(\xi)$ in $(3)_b$

Both transform into the same equation

$$4)\quad \frac{d}{d\xi}\left[(1-\xi^2)\frac{dy}{d\xi}\right] + \left[\mu^2 - \tfrac{1}{4} - \frac{m^2}{1-\xi^2}\right]y = 0$$

Taking $\mu - \frac{1}{2} = n$ gives solutions of $(2)_b$ in the form

$$5)\quad U^m_{(\alpha,\beta)} = \sqrt{\sin\alpha\cosh\beta}\; T^m_n(\cos\alpha)\left[A\,P^m_n(i\sinh\beta) + B\,Q^m_n(i\sinh\beta)\right]$$

Or (as in Whipple's transformation) let

$\xi = i\cot\alpha$ and $u(\alpha) = y(\xi)$ in $(3)_a$

$\xi = \tanh\beta$ and $v(\beta) = y(\xi)$ in $(3)_b$

Both transform into

$$6)\quad \frac{d}{d\xi}\left[(1-\xi^2)\frac{dy}{d\xi}\right] + \left[m^2 - 1/4 - \frac{\mu^2}{1-\xi^2}\right]y = 0$$ giving the forms

$$7)_a\quad U^m_{(\alpha,\beta)} = T^{\mu}_{m-1/2}(\tanh\beta)\left[A\,P^{\mu}_{m-1/2}(i\cot\alpha) + B\,Q^{\mu}_{m-1/2}(i\cot\alpha)\right]$$

or

$$7)_b\quad U^m_{(\alpha,\beta)} = \sqrt{\sin\alpha}\; T^{\mu}_{m-1/2}(\tanh\beta)\left[A\,T^m_{\mu-1/2}(\cos\alpha) + B\,T^m_{\mu-1/2}(-\cos\alpha)\right]$$

which are suitable when U is given on a hyperboloid ($\alpha = \alpha_1$)

If there are no sources on the x-axis

8)$_a$ $U^m \to 0$ like $\sin\alpha^{m+\frac{1}{2}}$ when $\alpha \to 0$ or π (i.e. like $\rho^{m+\frac{1}{2}}$ by IX (13)$_b$)

If there are no sources at infinity in the $x\rho$-plane

8)$_b$ $U^m \to 0$ like $e^{-(m+\frac{1}{2})\beta}$ as $\beta \to \infty$. IX (13)$_a$

In dealing with potentials of ellipsoids it is best to represent on the w-strip the z-half-plane with a cut from $z=0$ to $z=ic$ along the imaginary axis of z.

As shown by fig 1′ or 1″ the z′-plane is also represented on the same w-strip, where

9) $(z-x_0)z' = -c^2$ where $x_0 = c\sinh\beta_0$

The cut $\overline{OCO'}$ of the z-plane in fig 1 which generates both sides of a circular disc, is bent into a circular arc in fig 1′ while in fig 1″ it has become complementary to the cut of fig 1. With polar coordinates $z' = r'e^{i\theta'}$ the equation of the family of curves (β = constant) into which the family of ellipses inverts, is

9)$_a$ $$r' = \frac{c}{\sinh^2\beta - \sinh^2\beta_0}\left[-\sinh\beta_0\cos\theta' \pm \sqrt{1 - \frac{\cosh^2\beta_0}{\cosh^2\beta}\sin^2\theta'}\right]$$

The locus of $\beta = \beta_0$ is the infinite semicircle together with the curve $d'd_0D_0$ whose equation is

9)$_b$ $$r' = \frac{1}{2\cosh^2\beta_0}\left[\frac{\sinh\beta_0}{\cos\theta'} + \frac{\cos\theta'}{\sinh\beta_0}\right]$$

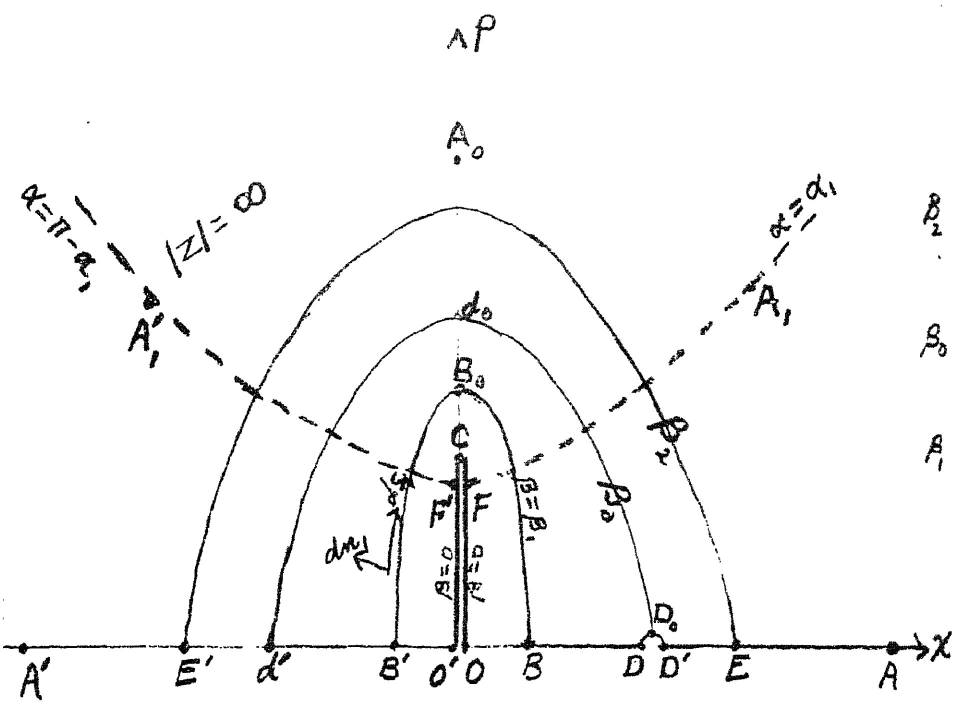


Fig 1 The z-halfplane $z = ic \sin\omega$ $(c>0)$

$\overline{OC} = c$

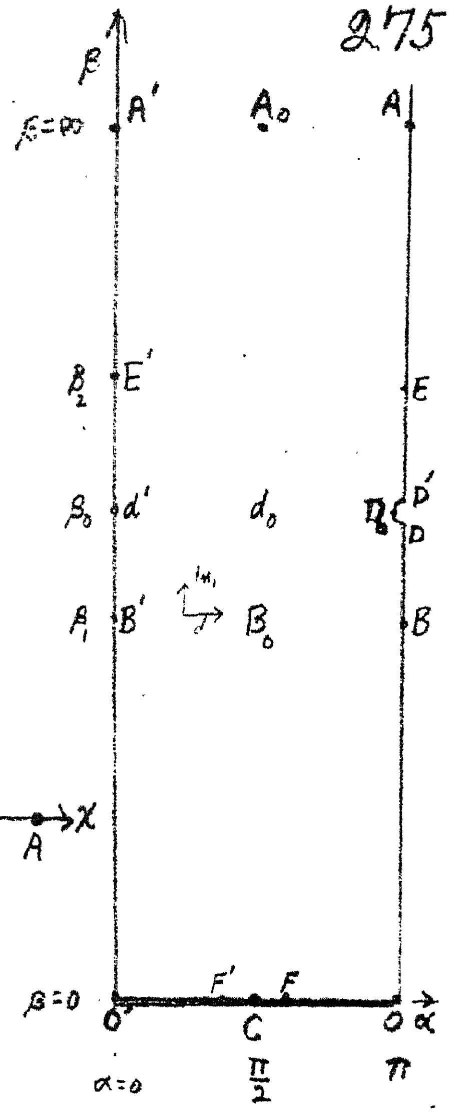


Fig 2 Semi-infinite strip of ω-plane

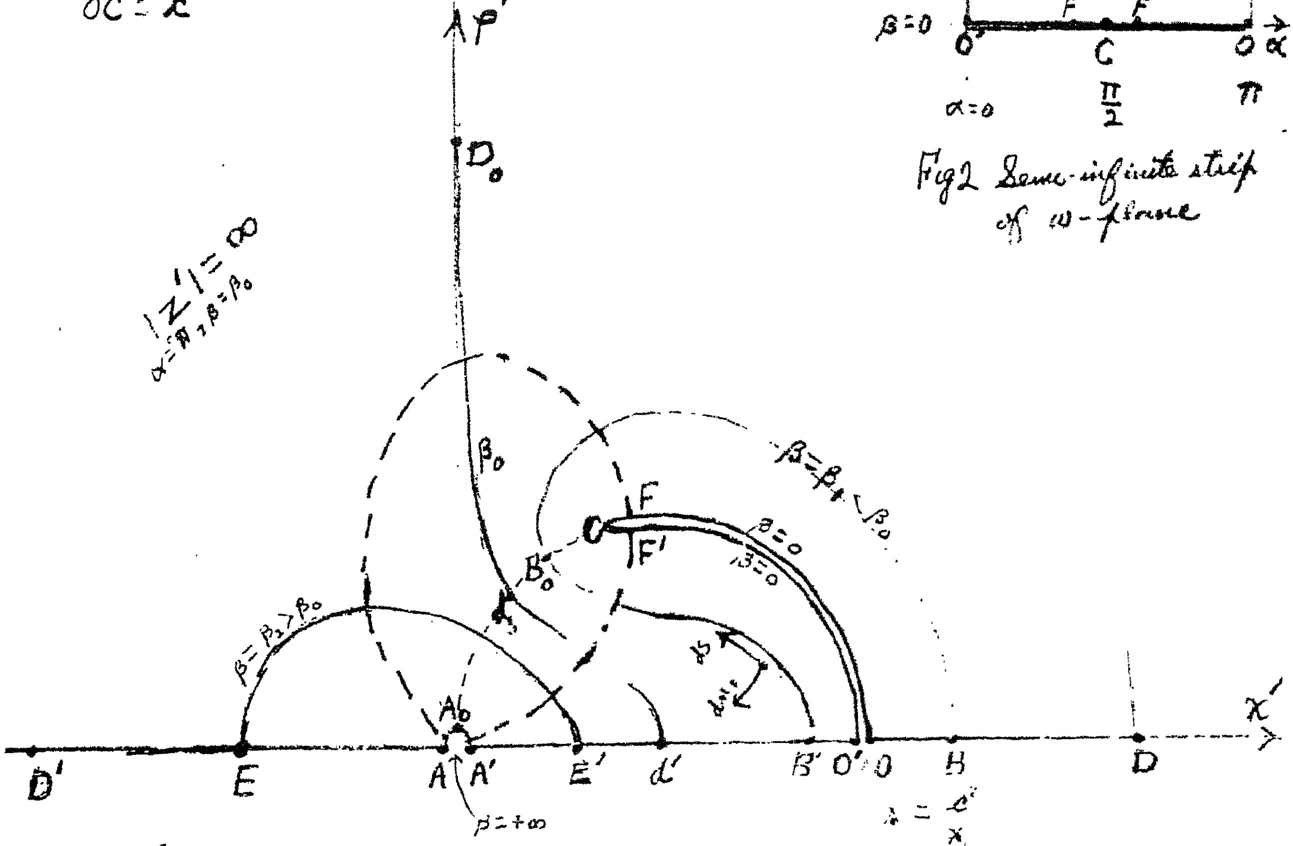
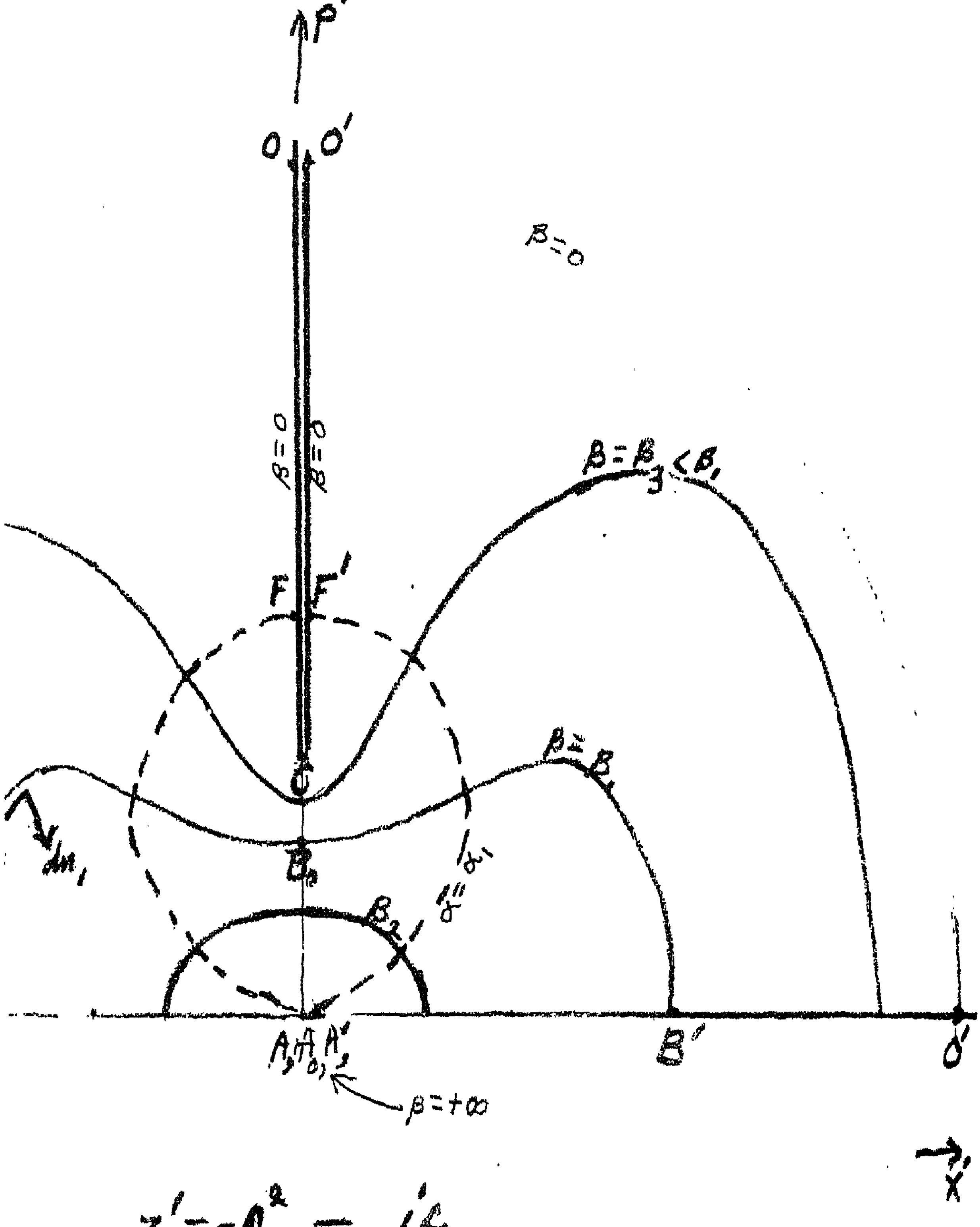


Fig 1' The z'-halfplane $(z - x_0) z' = -c^2$ for case $0 < x_0 = c \sin h \beta_0 < c$



$$z' = -\frac{a^2}{z} = \frac{ic}{\sin\omega}$$

(a) Potential given on an Oblate Spheroid or its Inversion

With solutions of type (5) the condition $(8)_a$ that the x axis be uncharged requires $n \geq m$, n being an integer, this being the only condition necessary as it was found to be also with spherical coordinates. Hence let

$$10)_a \quad \begin{cases} \mathcal{U}_n^m(\alpha) \equiv C_n^m \sqrt{\sin\alpha}\, T_n^m(\cos\alpha) \quad \text{where } C_n^m = \sqrt{(n+\frac{1}{2})\frac{(n-m)!}{(n+m)!}} \\ \text{so that} \\ \int_0^\pi \mathcal{U}_{n_1}^m(\alpha)\, \mathcal{U}_{n_2}^m(\alpha)\, d\alpha = \delta_{n_1 n_2} \end{cases}$$

$$10)_b \quad \mathcal{V}_n^{om}(\beta) \equiv \sqrt{\cosh\beta}\, . \, i e^{\frac{im\pi}{2}} Q_n^m(i\sinh\beta) \text{ and } \mathcal{V}_n^{om}(\beta) \equiv \sqrt{\cosh\beta}\, e^{-\frac{im\pi}{2}} P_n^m(i\sinh\beta)$$

These are real. The eq VI (7) becomes

$$11) \quad \cosh^2\beta\left[P_n^m(i\sinh\beta)\, Q_n^{\prime m}(i\sinh\beta) - P_n^{\prime m}(i\sinh\beta)\, Q_n^m(i\sinh\beta)\right] = (-1)^m \frac{(n+m)!}{(n-m)!}$$

which is equivalent to

$$12) \quad \gamma_n^m \equiv \mathcal{V}_n^{om}(\beta)\, \mathcal{V}_n^{\prime im}(\beta) - \mathcal{V}_n^{\prime om}(\beta)\, \mathcal{V}_n^{im}(\beta) = (-1)^m \frac{(n+m)!}{(n-m)!}$$

To show that $\mathcal{V}^o$ and $\mathcal{V}^i$ are real the equations of VI $(57)_a$ and $(57)_b$ may be written for this case $n \geq m = 0, 1, 2, 3, \text{------}$

$$13)_a \quad e^{-\frac{im\pi}{2}} P_n^m(i\sinh\beta) = \frac{(-1)^n \sqrt{\pi}\, 2^m \cosh^n\beta}{(n-m)!\, \Gamma(\frac{1}{2}-n)} F\left(\frac{m-n}{2}, \frac{-m-n}{2}, \frac{1}{2}-n; \operatorname{sech}^2\beta\right)$$

$$13)_b \quad i e^{\frac{im\pi}{2}} Q_n^m(i\sinh\beta) = \frac{(-1)^m \sqrt{\pi}\, (n+m)!}{2^{n+1} \cosh^{n+1}\beta\, \Gamma(n+\frac{3}{2})} F\left(\frac{n+m+1}{2}, \frac{n-m+1}{2}, n+\frac{3}{2}; \operatorname{sech}^2\beta\right)$$

These show that the sources of the potential (5), with P functions, are at $\beta=\infty$ where the Q functions are source-free. Hence the harmonics which would be called external with reference to any oblate spheroid of fig 1 (locus of $\beta=\beta_1$) must be

$$14)_a \quad \overset{\circ}{U}{}^m_n(\alpha,\beta) \equiv \mathcal{U}^m_n(\alpha)\frac{\overset{\circ}{V}{}^m_n(\beta)}{\overset{\circ}{V}{}^m_n(\beta_1)} \equiv C^m_n\sqrt{\frac{\sin\alpha\cosh\beta}{\cosh\beta_1}}\, T^m_n(\csc\alpha)\,\frac{Q^m_n(i\sinh\beta)}{Q^m_n(i\sinh\beta_1)}.$$

which obviously satisfy the conditions $(8)_a$ and $(8)_b$. The denominator $Q^m_n(i\sinh\beta_1)$ is never zero as shown by $(13)_b$.

For potentials of form (5) which are internal harmonics the factor $\sqrt{\sin\alpha}\, T^m_n(\csc\alpha)$ secures the source-free condition on the x axis. The construction of internal harmonics, requires a selection of those functions of β which will insure that there is neither simple nor double distribution on the cut OCO' of fig 1, where $\beta=0$. This is similar to the condition for internal harmonics of a toroid but the procedure is essentially different, since in that case the requirement of a source-free cut determined the eigen values and normal functions. In the present case these are already determined by making the solutions regular at the x axis.

The solution $(14)_a$ cannot continue to be harmonic when $\beta \to 0$, for it represents a simple distribution on the cut if $n-m$ is even, and a double distribution if $n-m$ is odd, since $T^m_n(\cos\alpha)$ has the same, or opposite sign at adjacent points on opposite sides of the cut, according as $n-m$ is even or odd. Adjacent points are represented by α and $\pi-\alpha$, that is $\cos\alpha$ and $-\cos\alpha$. Hence consider the potential

$$14)_b \quad U^{im}_n(\alpha,\beta) \equiv \mathcal{U}^m_n(\alpha)\,\frac{\mathcal{V}^{im}_n(\beta)}{\mathcal{V}^{im}_n(\beta_1)} \equiv C^m_n \sqrt{\frac{\sin\alpha\cosh\beta}{\cosh\beta_1}}\; T^m_n(\cos\alpha)\,\frac{P^m_n(i\sinh\beta)}{P^m_n(i\sinh\beta_1)}$$

Expressions for $e^{-\frac{in\pi}{2}} P^m_n$ according as $n-m$ is even or odd are

$$15)_a \quad e^{-\frac{in\pi}{2}} P^{-m}_n(i\sinh\beta) = \frac{2^m\,\Gamma(s+m+\frac{1}{2})}{\sqrt{\pi}\;\; s!}\cosh^m\beta\; F(-s,\, s+m+\tfrac{1}{2},\, \tfrac{1}{2};\, -\sinh^2\beta) \quad \text{if } n=m+2s$$

$$15)_b \quad e^{-\frac{in\pi}{2}} P^{-m}_n(i\sinh\beta) = \frac{2^{m+1}\,\Gamma(s+m+\frac{3}{2})}{\sqrt{\pi}\;\; s!}\sinh\beta\cosh^m\beta\; F(-s,\, s+m+\tfrac{3}{2},\, \tfrac{3}{2};\, -\sinh^2\beta) \quad \text{if } n=m+2s+1$$

In case $n-m$ is even the factor $T^m_n(\cos\alpha) = T^m_n(-\cos\alpha)$ in $(14)_b$ has the same value at adjacent points on opposite sides of the cut so the potential will be continuous and therefore will not represent a double distribution. Also eq $(1)_c$ with $(15)_a$ shows that its normal derivatives

vanish on each side of the cut. Therefore when $n-m$ is even, $(14)_e$ is an internal harmonic.

In case $n-m$ is odd, $\overset{m}{\underset{n}{T}}(\cos\alpha) = -\overset{m}{\underset{n}{T}}(-\cos\alpha)$, but in that case the factor $\overset{m}{\underset{n}{P}}$ vanishes, making the potential $(14)_e$ continuous at the cut, although $D_\beta \overset{im}{\underset{n}{V}}$ does not vanish when $\beta \to 0$. However, since the T-factor has opposite signs on opposite sides of the cut this makes the normal derivative continuous, and $(14)_e$ is an internal harmonic in all cases.

To show that the denominator in $(14)_e$ does not vanish, apply Euler's transformation to (15). This gives

$$15)_c \quad e^{-\frac{in\pi}{2}} \overset{m}{\underset{n}{P}}(i\sinh\beta) = \frac{2^m \Gamma(s+m+\frac{1}{2})}{\sqrt{\pi}\, s!\, \cosh_\beta^{1+2m+2s}} F(s+\tfrac{1}{2}, s+\tfrac{1}{2}+m, \tfrac{1}{2}; \tanh^2\beta) \quad \text{if } n = m+2s$$

$$15)_d \quad e^{-\frac{in\pi}{2}} \overset{m}{\underset{n}{P}}(i\sinh\beta) = \frac{2^{m+1}\Gamma(s+m+\frac{3}{2})\sinh\beta}{\sqrt{\pi}.s!\ \cosh_\beta^{3+2m+2s}} F(s+\tfrac{3}{2}, s+\tfrac{3}{2}+m, \tfrac{3}{2}; \tanh^2\beta) \quad \text{if } n = m+2s+1$$

The (reduced) potential given by $(14)_a$ and $(14)_e$ is continuous at $\beta=\beta_1$ where it has the value $\overset{m}{\underset{n}{u}}(\alpha)$. It is therefore due to a simple distribution on the curve $\beta=\beta_1$ whose (reduced) density is

$$16) \quad \frac{\overline{\sigma}^m_n(\alpha)}{h(\alpha,\beta_1)} = \overset{m}{\underset{n}{\lambda}}(\alpha_1)\, \overset{m}{\underset{n}{u}}(\alpha) \qquad \text{where}$$

$$17) \quad \overset{m}{\underset{n}{\lambda}}(\beta_1) \equiv \frac{\gamma^m_n}{4\pi\, \overset{im}{\underset{n}{V}}(\beta_1)\, \overset{om}{\underset{n}{V}}(\beta_1)} = \frac{(-1)^m \frac{(n+m)!}{(n-m)!}}{4\pi \cosh\beta_1\, \overset{m}{\underset{n}{P}}(i\sinh\beta_1)\, i\, \overset{m}{\underset{n}{Q}}(i\sinh\beta_1)}$$

The two forms of (14) are equivalent to the potential integral

$$18) \quad U_n^m(\alpha,\beta) = 2\int_0^\pi \frac{\bar{\sigma}_n^m(\alpha_1)}{h(\alpha_1,\beta_1)} Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))\,d\alpha_1 \quad \text{where}$$

$$19) \quad g(\alpha,\beta;\alpha_1,\beta_1) = 1 + \frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1} =$$

$$= \frac{\sin^2\alpha + \sinh^2\beta + \sin^2\alpha_1 + \sinh^2\beta_1 - 2\cos\alpha\sinh\beta\cos\alpha_1\sinh\beta_1}{2\sin\alpha\cosh\beta\sin\alpha_1\cosh\beta_1}$$

Substituting in (18) the expression (16) for σ/h and equating U to U^i or U^o gives the homogenous integral equation.

$$20)_a \quad 2\lambda_m^m(\beta_1)\int_0^\pi \mathcal{U}_n^m(\alpha_1) Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))\,d\alpha_1 = \mathcal{U}_m^m(\alpha)\frac{V_n^{om}(\beta)}{V_m^{om}(\beta_1)} \quad \text{if} \quad 0\le\beta_1\le\beta<\infty$$

that is,

$$= \mathcal{U}_m^m(\alpha)\frac{V_n^{im}(\beta)}{V_m^{im}(\beta_1)} \quad \text{if} \quad 0\le\beta\le\beta_1<\infty$$

$$20)_b \quad \int_0^\pi \sqrt{\sin\alpha_1}\, T_m^m(\cos\alpha_1) Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))\,d\alpha_1 =$$

$$= 2\pi(-1)^m \sqrt{\sin\alpha}\, T_m^m(\cos\alpha)\cdot\frac{(n-m)!}{(n+m)!}\sqrt{\cosh\beta\cosh\beta_1}\, P_m^m(i\sinh\beta_1)\, i\, Q_m^m(i\sinh\beta)$$

when $\beta_1 \le \beta$.

This integral is the Fourier coefficient in the series of normal functions representing $Q_{m-1/2}(g)$ as a

function of α. This series is therefore the canonical expansion of the symmetrical nucleus $Q_{m-1/2}(\varrho)$ or the addition-formula (as in IX (44)),

$$21)\quad Q_{m-1/2}\left(\frac{\sin^2\alpha+\sinh^2\beta+\sin^2\alpha_1+\sinh^2\beta_1-2\cos\alpha\sinh\beta\cos\alpha_1\sinh\beta_1}{2\sin\alpha\cosh\beta\sin\alpha_1\cosh\beta_1}\right)=$$

$$=(-1)^m 2\pi\sqrt{\sin\alpha\cosh\beta\sin\alpha_1\cosh\beta_1}\sum_{n=m}^{\infty}\left(n+\tfrac{1}{2}\right)\left[\frac{(n-m)!}{(n+m)!}\right]^2 T_n^m(\cos\alpha)\, T_n^m(\cos\alpha_1)\, P_n^m(i\sinh\beta)\, i\, Q_n^m(i\sinh\beta_1)$$

where $0 \leq \beta \leq \beta_1$

When $\beta=\beta_1=0$, both points are on the cut $\overline{OCO'}$ and this becomes

$$21)_a\quad Q_{m-1/2}\left(\tfrac{1}{2}\left(\frac{\sin\alpha}{\sin\alpha_1}+\frac{\sin\alpha_1}{\sin\alpha}\right)\right)=$$

$$=\frac{\pi\sqrt{\sin\alpha\sin\alpha_1}}{2^{2m}}\sum_{s=0}^{\infty}\left(2s+m+\tfrac{1}{2}\right)\left[\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+m+1)}\right]^2 T_{m+2s}^m(\cos\alpha)\, T_{m+2s}^m(\cos\alpha_1)$$ which is

valid for all values of α and α_1 between zero and π.

Another special case is $\alpha=\alpha_1=\pi/2$

$$21)_b\quad Q_{m-1/2}\left(\tfrac{1}{2}\left(\frac{\cosh\beta}{\cosh\beta_1}+\frac{\cosh\beta_1}{\cosh\beta}\right)\right)=$$

$$=\frac{(-1)^m\sqrt{\cosh\beta\cosh\beta_1}}{2^{2m-1}}\sum_{s=0}^{\infty}\left(2s+m+\tfrac{1}{2}\right)\left[\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+m+1)}\right]^2 P_{m+2s}^m(i\sinh\beta_1)\, i\, Q_{m+2s}^m(i\sinh\beta)$$

where $\beta_1 \leq \beta$.

The reduced potential with assigned values on the oblate spheroid (or on its inversion, the locus $\beta=\beta_1$) is given at all external points α,β where $\beta_1 \leq \beta \leq \infty$, by

$$22)\quad U^m(\alpha,\beta)=\sqrt{\frac{\sin\alpha\cosh\beta}{\cosh\beta_1}}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}T_n^m(\cos\alpha)\frac{Q_n^m(i\sinh\beta)}{Q_n^m(i\sinh\beta_1)}\int_0^{\pi}F(\alpha_1)\sqrt{\sin\alpha_1}\,T_n^m(\cos\alpha_1)\,d\alpha_1$$

The Q-functions are replaced by P-functions when $0 \leq \beta \leq \beta_1$. This reduces to $F(\alpha)$ when $\beta=\beta_1$ by the formula

$$23)\quad F(\alpha)=\sqrt{\sin\alpha}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}T_n^m(\cos\alpha)\int_0^{\pi}F(\alpha_1)\sqrt{\sin\alpha_1}\,T_n^m(\cos\alpha_1)\,d\alpha_1, \quad \text{for } 0<\alpha<\pi$$

In particular if $F(\alpha)$ is given on both sides of the cut the potential is given everywhere by the following limiting case of (22) when $\beta_1 \to 0$

$$24)\quad U^m(\alpha,\beta)= =\frac{(-1)^m\sqrt{\sin\alpha\cosh\beta}}{2^{m-1}\sqrt{\pi}}\sum_{n=m}^{\infty}(n+\tfrac{1}{2})\frac{(n-m)!}{(n+m)!}T_n^m(\cos\alpha)\frac{\Gamma(\frac{n-m}{2}+1)}{\Gamma(\frac{n+m+1}{2})}e^{i(n+1)\frac{\pi}{2}}Q_n^m(i\sinh\beta)\int_0^{\pi}F(\alpha_1)\sqrt{\sin\alpha_1}\,T_n^m(\cos\alpha_1)\,d\alpha_1$$

If $F(\alpha)$ is an even function of $\cos\alpha$, the only terms which remain in this series are those in which $n-m$ is even. The potential is then single valued at the cut and is that of a simple distribution on the cut. In fig 1 this generates both sides of a circular disc, which in

fig 1' has been bent into part of a spherical surface but in fig 1", a circular aperture in the plane $x=0$ has taken the place of the circular disc. This is the problem whose formal solution was given in Toroidal Coordinates

If $F(\alpha)$ is an odd function of $\cos\alpha$, only the terms of (24), with $n-m$ odd, survive. This is the potential of a double distribution on the cut corresponding to the magnetic potential of a current sheet or state of magnetic polarization at the cut.

(b) Potential given on a one-sheeted hyperboloid or its Inversion.

For this problem, instead of making the cut in the z plane as in fig 1, it is necessary to make it in a complementary manner from $\rho = c$ to $\rho = \infty$ as shown in fig 3. The z-half-plane thus cut is now represented upon the endless strip $0 < \alpha < \frac{\pi}{2}$, $-\infty < \beta < \infty$ of fig 4 which is half as wide and twice as long as that previously used in fig 2. This change is necessary in order that the locus $\alpha = \alpha_1 =$ constant, $-\infty < \beta < \infty$ may be the entire hyperbolic arc of fig 3 which generates the one-sheeted hyperboloid. The equations (1) to (9) remain valid, but inversions of this hyperboloid would be closed curves, that is, the dotted curve in fig 1' or 1'' instead of beginning and ending on a cut, now enclose the cut which is along A_0C. It is evident that this problem calls for a different type of analysis from the preceding and the solution required cannot be obtained from (24) by inversion since no actual ellipse inverts into a hyperbola. The exception is $\alpha = \pi/2$ when the hyperbola becomes both sides of the cut of fig 3 In that case the solution to be obtained will be

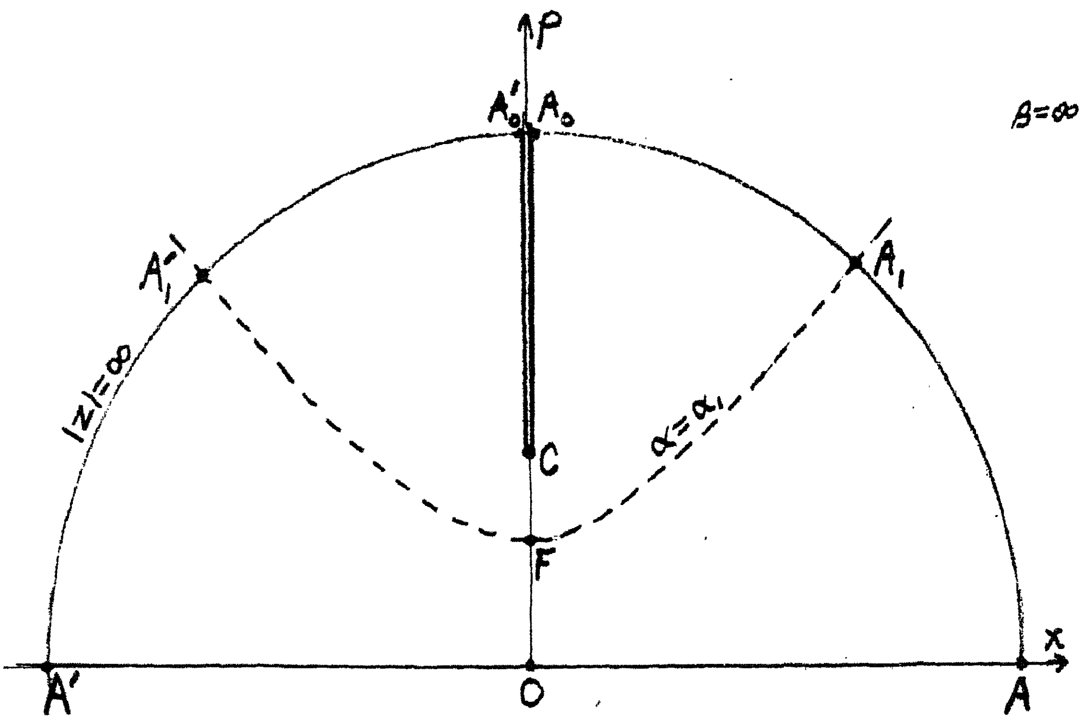


Fig 3 The z-halfplane

$z \equiv x + i\rho$

$z = ic\sin\omega$

$\omega \equiv \alpha + i\beta$

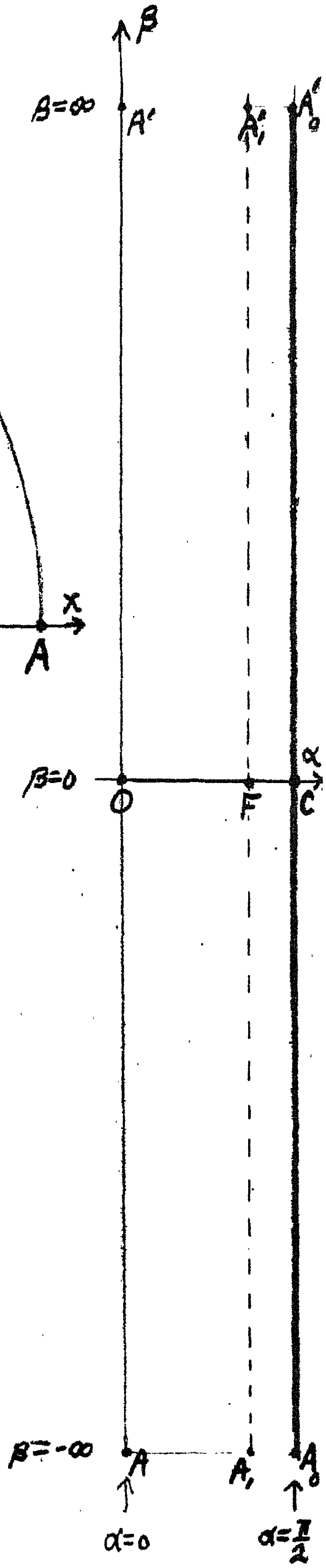


Fig 4 The ω-strip

equivalent to (but differing in form from) that obtainable from (24) as interpreted in fig 1."

For solutions of type $(7)_b$ we have the formulae

$$25)\quad \mathsf{T}^{\mu}_{m-1/2}(\tanh\beta) = \frac{\Gamma(\frac{1}{2}+m+\mu)\ e^{-\mu\beta}}{\Gamma(\frac{1}{2}+m-\mu)\ \Gamma(\mu+1)} F\left(\frac{1}{2}+m, \frac{1}{2}-m, \mu+1; \frac{1}{1+e^{2\beta}}\right)$$

Let

$$26)_a\quad \mathsf{T}^{\mu}_{m-1/2}(\tanh\beta) = C^{\mu}_{m}(\tanh\beta) - S^{\mu}_{m}(\tanh\beta)$$

$$26)_b\quad \mathsf{T}^{\mu}_{m-1/2}(-\tanh\beta) = C^{\mu}_{m}(\tanh\beta) + S^{\mu}_{m}(\tanh\beta)$$

where by VI (28), C and S are even and odd functions of $\tanh\beta$

$$26)_c\quad C^{\mu}_{m}(\tanh\beta) \equiv \frac{2^{\mu-1}}{\pi^{3/2}}\cos(\mu-m)\pi\, \mathrm{sech}^{\mu}\beta\, \Gamma\left(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}\right)\Gamma\left(\tfrac{1}{4}-\tfrac{m}{2}+\tfrac{\mu}{2}\right) F\left(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{1}{4}-\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{1}{2}; \tanh^2\beta\right)$$

$$26)_d\quad S^{\mu}_{m}(\tanh\beta) \equiv$$

$$\equiv \frac{2^{\mu}}{\pi^{3/2}}\cos(\mu-m)\pi.\ \tanh\beta\, \mathrm{sech}^{\mu}\beta\, \Gamma\left(\tfrac{3}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}\right)\Gamma\left(\tfrac{3}{4}-\tfrac{m}{2}+\tfrac{\mu}{2}\right) F\left(\tfrac{3}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{3}{4}-\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{3}{2}; \tanh^2\beta\right)$$

These are valid for unrestricted m and μ so that, replacing $\tanh\beta$ by $\cos\alpha$ and interchanging m and μ we obtain

$$27)\quad \mathsf{T}^{m}_{\mu-1/2}(\cos\alpha) = \frac{\Gamma(\frac{1}{2}+m+\mu)}{\Gamma(\frac{1}{2}-m+\mu)}\frac{\tan^m \alpha/2}{m!} F\left(\tfrac{1}{2}+\mu, \tfrac{1}{2}-\mu, m+1; \sin^2\tfrac{\alpha}{2}\right)$$

$$28)_a\quad \mathsf{T}^{m}_{\mu-1/2}(\cos\alpha) = C^{m}_{\mu}(\cos\alpha) - S^{m}_{\mu}(\cos\alpha)$$

$$28)_b\quad \mathsf{T}^{m}_{\mu-1/2}(-\cos\alpha) = C^{m}_{\mu}(\cos\alpha) + S^{m}_{\mu}(\cos\alpha)$$

where

$$28)_c \quad C_\mu^m(\cos\alpha) =$$
$$= \frac{2^{m-1}}{\pi^{3/2}} \cos(\mu-m)\pi .\sin^m\alpha \, \Gamma(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{\mu}{2})\Gamma(\tfrac{1}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}) F(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{1}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}, \tfrac{1}{2}; \cos^2\alpha)$$

$$28)_d \quad S_\mu^m(\cos\alpha) =$$
$$= \frac{2^m}{\pi^{3/2}} \cos(\mu-m)\pi .\cos\alpha \sin^m\alpha \, \Gamma(\tfrac{3}{4}+\tfrac{m}{2}+\tfrac{\mu}{2})\Gamma(\tfrac{3}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}) F(\tfrac{3}{4}+\tfrac{m}{2}+\tfrac{\mu}{2}, \tfrac{3}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}, \tfrac{3}{2}; \cos^2\alpha)$$

From these we find, when $\alpha \to \pi/2$, $\cos\alpha \to 0$,

$$29)_a \quad C_m^m(0) = \frac{(-1)^m 2^{m-1}}{\pi^{3/2}} \cos\mu\pi \, \Gamma(\tfrac{1}{4}+\tfrac{m}{2}+\tfrac{\mu}{2})\Gamma(\tfrac{1}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}) \text{ and } D_\alpha C_\mu^m(\cos\alpha) \to 0$$

$$29)_b \quad S_\mu^m(0) = 0 \text{ and } D_\alpha S_\mu^m(\cos\alpha) \to \frac{(-1)^{m+1} 2^m}{\pi^{3/2}} \cos\mu\pi \, \Gamma(\tfrac{3}{4}+\tfrac{m}{2}+\tfrac{\mu}{2})\Gamma(\tfrac{3}{4}+\tfrac{m}{2}-\tfrac{\mu}{2}).$$

whence

$$30) \quad \sqrt{\sin\alpha}\left\{ S_\mu^m(\cos\alpha) D_\alpha\left(\sqrt{\sin\alpha}\, C_\mu^m(\cos\alpha)\right) - C_\mu^m(\cos\alpha) D_\alpha\left(\sqrt{\sin\alpha}\, S_\mu^m(\cos\alpha)\right)\right. = \frac{1}{\Gamma(\tfrac{1}{2}-m-\mu)\Gamma(\tfrac{1}{2}-m+\mu)}$$

When $0 \leq \alpha \leq \pi/2$ these four functions of α are even integral functions of μ. When $\alpha \to 0$, the only one which is finite with its derivative is $T_{\mu-1/2}^m(\cos\alpha)$. Hence in the region $0 \leq \alpha < \alpha_1$ "external" to the hyperbola, $\alpha = \alpha_1$, the condition $(8)_a$ for no sources on the x axis requires that the solutions $(7)_e$ be of the form

$$U^{om}(\alpha,\beta) = \sqrt{\sin\alpha}\; T_{\mu-1/2}^m(\cos\alpha)\left[A\, T_{m-1/2}^\mu(\tanh\beta) + B\, T_{m-1/2}^\mu(-\tanh\beta)\right]$$

$$= \sqrt{\sin\alpha}\; T_{\mu-1/2}^m(\cos\alpha)\left[A' C_m^\mu(\tanh\beta) + B' S_m^\mu(\tanh\beta)\right]$$

The parameter μ have nothing to do with this condition. "Internal" harmonics (for the region $\alpha_1 \leq \alpha < \pi/2$) must have neither simple nor double distributions at the cut $\alpha = \pi/2$, A_0CA_0' of fig 3. Adjacent points on opposite sides of the cut correspond to equal and opposite values of β. Hence the even function of β, $C^{\mu}_{m}(\tanh\beta)$, must have the factor $\sqrt{\sin\alpha}\, C^{m}_{\mu}(\cos\alpha)$ whose derivative vanishes with $\cos\alpha$. The odd functions, $S^{\mu}_{m}(\tanh\beta)$ must be associated with the factor $S^{m}_{\mu}(\cos\alpha)$ which vanishes at the cut. Therefore the internal harmonics are of the form

$$U^{im}_{(\alpha,\beta)} = \sqrt{\sin\alpha}\left[A\, C^{m}_{\mu}(\cos\alpha)\, C^{\mu}_{m}(\tanh\beta) + B\, S^{m}_{\mu}(\cos\alpha)\, S^{\mu}_{m}(\tanh\beta)\right]$$

From IX (15) it is evident that any potential U^{m}, with external expression U^{om} and internal U^{im} which is continuous at $\alpha = \alpha_1$ and satisfies the condition (8)$_a$ for no sources on the x axis and (8)$_b$ for none at infinity must be the (reduced) potential of a simple distribution on the curve $\alpha = \alpha_1$ whose (reduced) density is given by

$$31)_a \quad \frac{4\pi\bar{\sigma}(\beta)}{h(\alpha_1,\beta)} = \left(D_\alpha U^{m}\right)_{\alpha=\alpha_1-0} - \left(D_\alpha U^{m}\right)_{\alpha=\alpha_1+0}$$

If this density vanishes when $\beta \to \pm\infty$ or if $\frac{\bar{\sigma}}{h}$ becomes infinite like $e^{\delta|\beta|}$ when $\beta \to \pm\infty$, where $\delta < m + 1/2$, then the potential $U^{m}_{(\alpha,\beta)}$ is given at all finite points (α,β) by the absolutely convergent integral

31)$_b$ $\quad U^m_{(\alpha,\beta)} = 2\int_{-\infty}^{\infty} \frac{\bar{\sigma}(\beta_1)}{h(\alpha_1,\beta_1)} Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))d\beta_1$ $\quad$ where g is given by (19)

This may be applied to the following continuous potentials which have no sources on the x axis or at the cut.

32)$_a$ $\quad U^{om}_c(\alpha,\beta) = \sqrt{\sin\alpha\sin\alpha_1}\, T^m_{\mu-1/2}(\cos\alpha)\, C^m_\mu(\cos\alpha_1)\, C^\mu_m(\tanh\beta)$ $\quad$ where $0\le\alpha\le\alpha_1$

32)$_b$ $\quad U^{im}_c(\alpha,\beta) = \sqrt{\sin\alpha\sin\alpha_1}\, T^m_{\mu-1/2}(\cos\alpha_1)\, C^m_\mu(\cos\alpha)\, C^\mu_m(\tanh\beta)$ $\quad$ where $\alpha_1\le\alpha<\frac{\pi}{2}$

then by (30) and (31)$_a$

32)$_c$ $\quad \frac{4\pi\bar{\sigma}_c(\beta)}{h(\alpha_1,\beta)} = \frac{C^\mu_m(\tanh\beta)}{\Gamma(\frac{1}{2}-m-\mu)\Gamma(\frac{1}{2}-m+\mu)}$

and similarly the odd functions of β

33)$_a$ $\quad U^{om}_s(\alpha,\beta) = \sqrt{\sin\alpha\sin\alpha_1}\, T^m_{\mu-1/2}(\cos\alpha)\, S^m_\mu(\cos\alpha_1)\, S^\mu_m(\tanh\beta)$ $\quad$ where $0\le\alpha\le\alpha_1$

33)$_b$ $\quad U^{im}_s(\alpha,\beta) = \sqrt{\sin\alpha\sin\alpha_1}\, T^m(\cos\alpha_1)\, S^m_\mu(\cos\alpha)\, S^\mu_m(\tanh\beta)$ $\quad$ where $\alpha_1\le\alpha<\frac{\pi}{2}$

These have the density given by

33)$_c$ $\quad \frac{4\pi\bar{\sigma}_s(\beta)}{h(\alpha_1,\beta)} = \frac{S^\mu_m(\tanh\beta)}{\Gamma(\frac{1}{2}-m-\mu)\Gamma(\frac{1}{2}-m+\mu)}$

If the parameter μ $(\equiv\mu_1+i\mu_2)$ is represented by a point in the strip of the μ-plane, $-(m+\frac{1}{2})<\mu_1<m+\frac{1}{2}$, the potential integral (31)$_b$ will converge with the densities given by (32)$_c$ and (33)$_c$. This integral then becomes a homogenous integral equation which is

multiplied by C^{μ}_{m} and S^{μ}_{m}.

[illegible] $Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1))$ [illegible] $=$

$= [illegible]\sqrt{\sin\alpha\,\sin\alpha_1}\,\Gamma(\tfrac{1}{2}-m-\mu)\,\Gamma(\tfrac{1}{2}-m+\mu)$ [illegible] $C^{\mu}_{m}(\tanh\beta_1)$

and a similar equation when the C-functions are all replaced by S-functions with the same variables. This integral and its mate are the transforms for the integral representation of $Q_{m-1/2}(g)$ as a function of β which is given in VIII (31).

Therefore, when $0 \leq \alpha \leq \alpha_1$

(35) $Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1)) =$

$$= \frac{[illegible]\sqrt{\sin\alpha\,\sin\alpha_1}}{i}\int_{\mu_1-i\infty}^{\mu_1+i\infty}\frac{\mu\,\Gamma^2(\tfrac{1}{2}-m-\mu)}{\cos\mu\pi}\,\Gamma(\ldots)\Big[C^{\mu}_{\mu}(\cos\alpha_1)\,C^{\mu}_{m}(\tanh\beta)\,C^{\mu}_{m}(\tanh\beta_1)$$

where $m-\frac{1}{2} < \mu_1 < m+\frac{1}{2}$, $\qquad - S^{\mu}_{\mu}(\cos\alpha_1)\,S^{\mu}_{m}(\tanh\beta)\,S^{\mu}_{m}(\tanh\beta_1)\Big]$

These integrals may be transformed into the series (21) by use of Whipple's relations. If α and α_1 are interchanged in (35) it becomes valid for $\alpha_1 \leq \alpha \leq \pi/2$.

If there is a single distribution with (reduced) density $\bar{\sigma}(\beta)$ on the hyperboloid (or locus $\alpha = \alpha_1$) and if the

total charge is finite, and the (reduced) potential is $f(\beta)$ when $\alpha = \alpha_1$ then $f(\beta)$ and $\bar{\sigma}(\beta)/h(\alpha,\beta)$ will vanish like $e^{-(m+\frac{1}{2})|\beta|}$ when $\beta \to \pm\infty$. Hence using (35) and the potential integral $(31)_b$ gives for $0 \leq \alpha \leq \alpha_1$

36) $$U^{om}_{(\alpha,\beta)} = \frac{2(-1)^m \pi \sqrt{\sin\alpha \sin\alpha_1}}{i} \int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \frac{\mu \Gamma^2(\frac{1}{2}-m-\mu)}{\cos\mu\pi} T^m_{\mu-1/2}(\cos\alpha)\, d\mu \int_{-\infty}^{\infty} \frac{\bar{\sigma}(\beta_1)}{h(\alpha_1,\beta_1)} \Big[C^m_\mu(\cos\alpha_1)\, C^\mu_m(\tanh\beta)\, C^\mu_m(\tanh\beta_1)$$

$m - \frac{1}{2} < \mu_1 < m + \frac{1}{2}$ $$- S^m_\mu(\cos\alpha_1)\, S^\mu_m(\tanh\beta)\, S^\mu_m(\tanh\beta_1)\Big] d\beta_1$$

The potential $U^{im}_{(\alpha,\beta)}$ for the remaining region $\alpha_1 \leq \alpha < \frac{\pi}{2}$ is the same with α and α_1 interchanged.

The formal expression for the potential as an integral expressed in terms of its assigned values at $\alpha = \alpha_1$ is not so symmetrical as (36) (in the variables α and α_1). Placing $\alpha = \alpha_1$ in (36) and $U^{om}_{(\alpha_1,\beta)} = U^{im}_{(\alpha_1,\beta)} = f(\beta)$ and comparing this equation with formula $(31)_c$ of section VIII gives the transforms of the function $f(\beta)$ in terms of those of $\bar{\sigma}/h$

$37)_a$ $$C^m_\mu(\cos\alpha_1) G(\mu) \int_{-\infty}^{\infty} \frac{\bar{\sigma}(\beta_1)}{h(\alpha_1,\beta_1)} C^\mu_m(\tanh\beta_1)\, d\beta_1 = \int_{-\infty}^{\infty} f(\beta_1)\, C^\mu_m(\tanh\beta_1)\, d\beta_1$$

$37)_b$ $$S^m_\mu(\cos\alpha_1) G(\mu) \int_{-\infty}^{\infty} \frac{\bar{\sigma}(\beta_1)}{h} S^\mu_m(\tanh\beta_1)\, d\beta_1 = \int_{-\infty}^{\infty} f(\beta_1)\, S^\mu_m(\tanh\beta_1)\, d\beta_1$$

where

$37)_c$ $$G(\mu) \equiv \frac{4\pi^3 \sin\alpha_1\, T^m_{\mu-1/2}(\cos\alpha_1)}{\cos^2\mu\pi\, \Gamma(\frac{1}{2}-m-\mu)\Gamma(\frac{1}{2}+m-\mu)}$$

Using these in (36) gives

38)$_a$ $U^{om}_{(\alpha,\beta)} =$

$$= \frac{1}{2\pi i}\sqrt{\frac{\sin\alpha}{\sin\alpha_1}}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu\,\Gamma\left(\tfrac{1}{2}-m-\mu\right)\Gamma\left(\tfrac{1}{2}+m-\mu\right)\frac{T^{-m}_{\mu-1/2}(\cos\alpha)}{T^{m}_{\mu-1/2}(\cos\alpha_1)}\,d\mu\int_{-\infty}^{\infty} f(\beta_1)\Big[C^{\mu}_{m}(\tanh\beta)\,C^{\mu}_{m}(\tanh\beta_1)$$

$$- S^{\mu}_{m}(\tanh\beta)\,S^{\mu}_{m}(\tanh\beta_1)\Big]\,d\beta_1$$

$m - \frac{1}{2} < \mu_1 < m + \frac{1}{2}$

which is valid for $0 \leq \alpha \leq \alpha_1$ and $-\infty < \beta < \infty$

In the remaining region where $\alpha_1 \leq \alpha < \frac{\pi}{2}$, $-\infty < \beta < \infty$ we find

38)$_b$ $U^{im}_{(\alpha,\beta)} =$

$$= \frac{1}{2\pi i}\sqrt{\frac{\sin\alpha}{\sin\alpha_1}}\int_{\mu_1 - i\infty}^{\mu_1 + i\infty} \mu\,\Gamma\left(\tfrac{1}{2}-m-\mu\right)\Gamma\left(\tfrac{1}{2}+m-\mu\right)d\mu\int_{-\infty}^{\infty} f(\beta_1)\Big[\frac{C^{m}_{\mu}(\cos\alpha)}{C^{m}_{\mu}(\cos\alpha_1)}\,C^{\mu}_{m}(\tanh\beta)\,C^{\mu}_{m}(\tanh\beta_1)$$

$$- \frac{S^{m}_{\mu}(\cos\alpha)}{S^{m}_{\mu}(\cos\alpha_1)}\,S^{\mu}_{m}(\tanh\beta)\,S^{\mu}_{m}(\tanh\beta_1)\Big]\,d\beta_1$$

$m - \frac{1}{2} < \mu_1 < m + \frac{1}{2}$

Both of these become, when $\alpha = \alpha_1$, the integral representation, given in $\boxed{\text{VIII}}$ $(31)_2$, of a function $f(\beta)$ whose positive constant δ exceeds $m - \frac{1}{2}$.

(6) Annular Coordinates and their Inversion

Toroidal and oblate spheroidal coordinates have the same kind of cut in the z-half-plane, along the ρ axis from zero to $\rho = c$. This is bent by inversion into a circular cut in the z' plane which begins perpendicularly on the x axis and stops at some point in the half-plane of z'. The cut generates both sides of a circular disc, which inverts into both sides of part of a sphere. These two coordinate systems are the two limiting cases of annular coordinates, which arise by cutting the z half plane as before along the ρ axis from $\rho = 0$ to $\rho = a_1 > 0$. The half-plane thus cut is represented upon the rectangle of the u-plane where $u \equiv \alpha + i\beta$, $-2K < \alpha < 2K$, $0 < \beta < 2K'$ by the equation

$$1) \qquad z = -a_1 \frac{\operatorname{cn} u/2}{\operatorname{sn} u/2} = i a_1 \operatorname{dn}\left(\frac{u - 2iK'}{2}\right) \quad \text{where } \kappa = \sqrt{1 - \frac{a_2^2}{a_1^2}}, \quad \kappa' = \frac{a_2}{a_1}$$

or

$$2) \qquad x = -a_1 \frac{\operatorname{sn}\alpha/2 \operatorname{cn}\alpha/2 \operatorname{cn}\beta/2 \operatorname{dn}\beta/2}{1 - \operatorname{cn}^2\alpha/2 \operatorname{cn}^2\beta/2} \quad \text{and} \quad \rho = a_1 \frac{\operatorname{dn}\alpha/2 \operatorname{sn}\beta/2}{1 - \operatorname{cn}^2\alpha/2 \operatorname{cn}^2\beta/2}$$

where the modulus of functions of u, α or $i\beta$ is κ, but that of functions of β is the complementary modulus κ'.

The conformance between the z-half-plane of fig 1 and and the u-rectangle of fig 3 is shown by the lettering. The locus $\beta = 2\bar{K}'$ is both sides of a circular annulus or washer in the plane $x=0$, corresponding to $a_2 < \rho < a_1$. This is the heavy double line $\overline{A_2 A_1 A_2'}$, the remainder of the cut, the double dotted line $\overline{OA_2}$ and $\overline{O'A_2'}$ is represented at opposite ends of the u-rectangle. It is the locus $\alpha = 2\bar{K}$ and $\alpha = -2\bar{K}$. Every locus $\beta = \beta_1$ is a closed curve $\overline{B_2 B_1 B_2'}$ surrounding the annulus, beginning and ending on the dotted cut. The orthogonal family of curves $\alpha = \alpha_1$ begin normally on the x-axis and end normally on one side of the annulus. In particular the locus $\alpha = -\bar{K}$ is the quartercircle $\overline{DE}$ whose radius is $\sqrt{a_1 a_2}$, that of $\alpha = +\bar{K}$, the quartercircle $D'E'$.

The following discussion, being made in terms of α and β, may also be interpreted upon the z'-plane by the inversion formula

1)' $z' = \frac{c(z-c)}{z+c}$ which is shown in fig 2 for the case $c>0$.

The annulus is here bent into a circular arc, both ends of which are in the z'-half-plane. This generates a spherical surface with two coaxial holes.

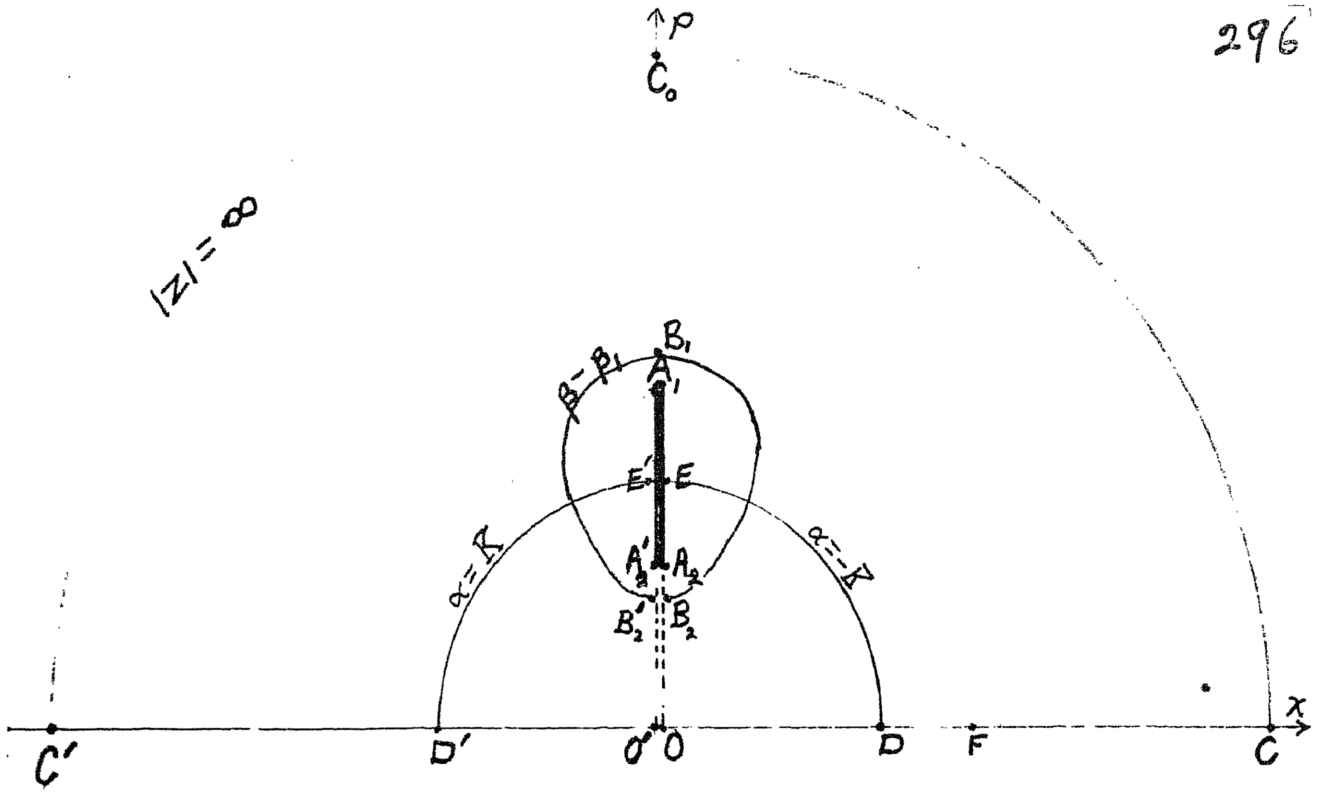


Fig 1 z-plane

$z = -a_1 \dfrac{cn\frac{u}{2}}{sn\frac{u}{2}}$ $\quad k' = \dfrac{a_2}{a_1}$

$\overline{OA_2} = a_2$, $\overline{OA_1} = a_1$

$\overline{OD} = \sqrt{a_1 a_2}$

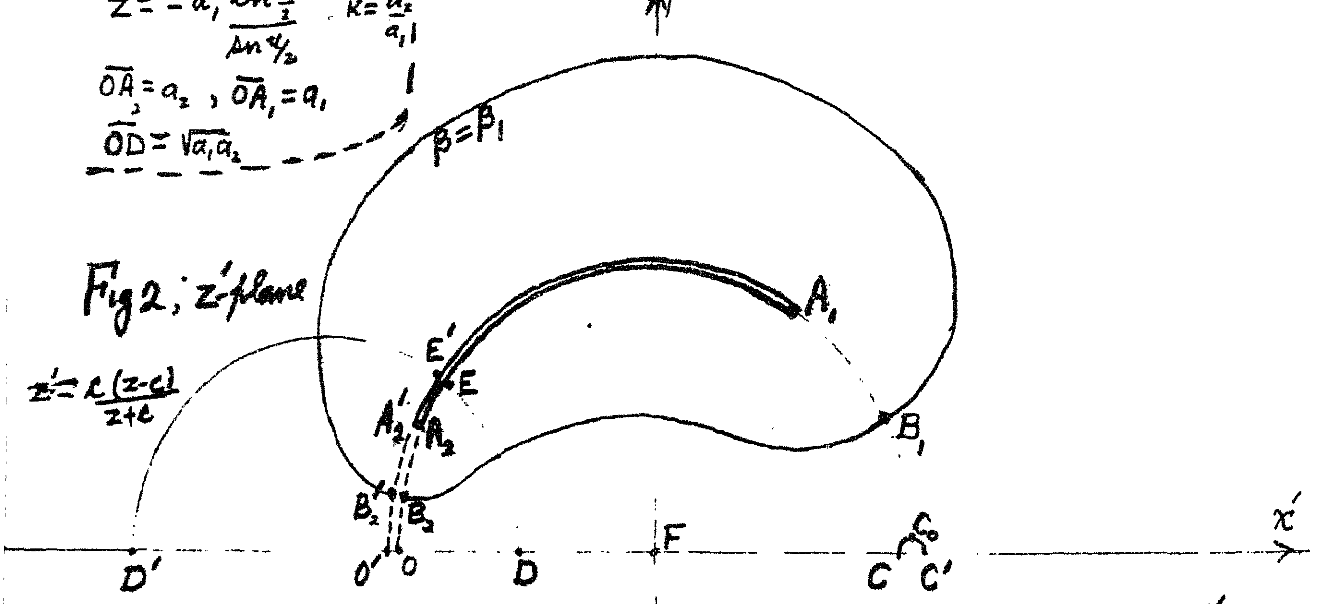


Fig 2; z'-plane

$z' = \dfrac{c(z-c)}{z+c}$

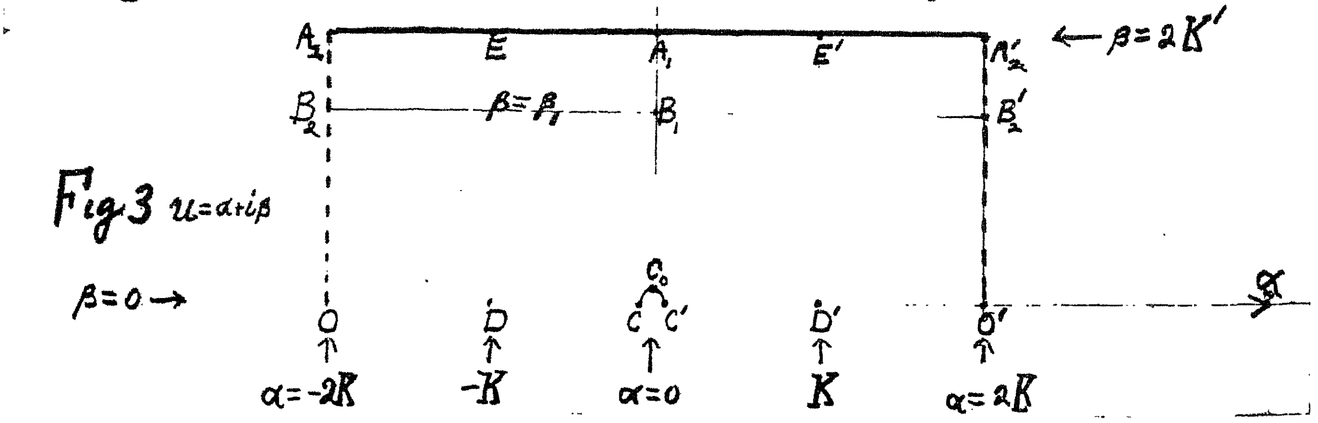


Fig 3 $u = \alpha + i\beta$

The group of transformation (1) and (1') is a case of Wangerin's transformation $z = f(u)$ where $\left(\frac{dz}{du}\right)^2 = R(z) =$ a quartic with four complex roots, the points A_1, A_2 and their conjugates. When A_1 and A_2 coincide it reduces to toroidal coordinates, but when A_2 and O coincide $a_2 \to 0$ it becomes an inversion of oblate spheroidal coordinates.

From eq(1), $\frac{dz}{du} = \frac{a_1}{2} \frac{dn\frac{u}{2}}{sn^2\frac{u}{2}}$ so that

$$\left(\frac{dz}{du}\right)^2 = \frac{a_1^2}{4} \frac{dn^2 u/2}{sn^4 u/2} = \frac{1}{4a_1^2}(z^2+a_1^2)(z^2+a_2^2) \quad \text{and}$$

$$3) \quad \frac{1}{h^2} = \left|\frac{dz}{du}\right|^2 = \frac{a_1^2}{4} \frac{dn^2\alpha/2\, cn^2\beta/2\, dn^2\beta/2 + \kappa^4 sn^2\alpha/2\, cn^2\alpha/2\, sn^2\beta/2}{(1 - cn^2\alpha/2\, cn^2\beta/2)^2}$$

Hence we may write (by (2) and (3))

$$4) \quad \frac{1}{\rho^2 h^2} = p(\alpha) - q(i\beta) \quad \text{where}$$

$$4)_a \quad p(\alpha) \equiv \frac{\kappa^2}{4}\frac{(1-dn\alpha)}{(1+dn\alpha)} = \frac{\kappa^4}{4}\frac{sn^2\frac{\alpha}{2}\, cn^2\alpha/2}{dn^2\alpha/2} = \text{a positive real}$$

$$4)_b \quad q(i\beta) \equiv \frac{\kappa^2}{4}\left[\frac{1-dn\, i(\beta-2K')}{1+dn\, i(\beta-2K')}\right] = \frac{\kappa^2}{4}\left(\frac{1+dn\, i\beta}{1-dn\, i\beta}\right) = -\frac{cn^2\beta/2\, dn^2\beta/2}{4\, sn^2\beta/2} = \text{a negative real.}$$

Equation (3) may also be written

$$5) \quad \frac{1}{h^2} = \frac{1}{4a_1^2}\sqrt{\left[(x^2+\rho^2)^2+(a_1^2+a_2^2)(x^2-\rho^2)+a_1^2a_2^2\right]^2+4(a_1^2-a_2^2)^2x^2\rho^2}$$

From (2), $(4)_a$ and $(4)_b$ we find

6) $$4pq = -\frac{K^4 x^2}{4\rho^2}$$ which being added to the square of (4) gives

$$p+q = \frac{\pm 1}{4a_1^2\rho^2}\sqrt{\left(\frac{4a_1^2}{h^2}\right)^2 - 4(a_1^2-a_2^2)^2x^2\rho^2}$$ or by (5)

7) $$p(\alpha)+q(i\beta) = \frac{-\left[(x^2+\rho^2)^2+(a_1^2+a_2^2)(x^2-\rho^2)+a_1^2a_2^2\right]}{4a_1^2\rho^2}$$

the negative sign being necessary because $q(i\beta)$, which is never positive, becomes $-\infty$ on the x-axis ($\rho=0$). Also when $x\to\pm 0$ while $a_2<\rho<a_1$, $\beta\to\mp 2K'$ and $q\to 0$ so (7) becomes $p(\alpha)=\frac{(a_1^2-\rho^2)(\rho^2-a_2^2)}{4a_1^2\rho^2}$ which is positive as required by $(4)_a$.

Adding (4) and (7) gives as the equation of the family of meridian curves, α = constant,

8) $$(x^2+\rho^2)^2+\left[a_1^2+a_2^2-\frac{(a_1^2-a_2^2)^2}{4a_1^2p(\alpha)}\right]x^2-\left[a_1^2+a_2^2-4a_1^2p(\alpha)\right]\rho^2+a_1^2a_2^2=0$$

Subtracting (4) and (7) shows that the equation of the family orthogonal to this is the same with $p(\alpha)$ replaced by $q(i\beta)$. Both are families of confocal cyclids.

Euler's equation, $\left(D_x^2 + D_\rho^2 + \frac{1/4 - m^2}{\rho^2}\right) V^m = 0$ becomes

9) $\left[D_\alpha^2 + (1/4 - m^2)\, p(\alpha) + D_\beta^2 - (\tfrac{1}{4} - m^2)\, q(i\beta)\right] V^m_{(\alpha,\beta)} = 0$

This has solutions of the form

10) $V^m_{(\alpha,\beta)} = u(\alpha) \cdot v(i\beta)$ where u and v are real function of their respective real and pure imaginary arguments which satisfy the ordinary differential equations

11)$_a$ $u''(\alpha) + \left[(1/4 - m^2)\, p(\alpha) + \nu\right] u(\alpha) = 0,$

11)$_b$ $v''(i\beta) + \left[(1/4 - m^2)\, q(i\beta) + \nu\right] v(i\beta) = 0.$

These may both be written in the same form

12)$_a$ $\begin{cases} \dfrac{d^2}{d\omega^2} u(\omega) + \left[(\tfrac{1}{4} - m^2) \dfrac{K^4 sn^2 \frac{\omega}{2}\, cn^2 \frac{\omega}{2}}{4\, dn^2 \omega/2} + \nu\right] u(\omega) = 0 \\ \text{where } \omega = \alpha = \text{a real independent variable.} \end{cases}$

12)$_b$ $\begin{cases} \dfrac{d^2}{d\omega^2} v(\omega) + \left[(\tfrac{1}{4} - m^2) \dfrac{K^4 sn^2 \frac{\omega}{2}\, cn^2 \frac{\omega}{2}}{4\, dn^2 \omega/2} + \nu\right] v(\omega) = 0 \\ \text{where } \omega = i(\beta - 2B') = \text{a pure imaginary variable.} \end{cases}$

Let

13) $u = dn^{m+\frac{1}{2}} \frac{\omega}{2} \cdot y$, then

14) $\dfrac{d^2 y}{d\omega^2} - (m + \tfrac{1}{2}) \dfrac{K^2 sn\,\omega/2\; cn\,\omega/2}{dn\,\omega/2} \cdot \dfrac{dy}{d\omega} + \left[(2\, sn^2 \tfrac{\omega}{2} - 1)(m + \tfrac{1}{2}) \tfrac{K^2}{4} + \nu\right] y = 0.$

Or letting

15) $z \equiv sn^2 \omega/2$ and $a \equiv 1/K^2$, this becomes

16) $\dfrac{d^2 y}{dz^2} + \left[\dfrac{1/2}{z} + \dfrac{1/2}{z-1} + \dfrac{m+1}{z-a}\right] \dfrac{dy}{dz} + \dfrac{(m+\frac{1}{2})\frac{1}{2} z + a\nu - \frac{1}{4}(m+\frac{1}{2})}{z(z-1)(z-a)}\, y = 0$

This is an equation of the normal form VII (3) satisfied by Heun's function of z, with parameters

17) $a = 1/k^2$, $b = a\nu - \frac{1}{4}(m+\frac{1}{2})$, $\alpha = m+\frac{1}{2}$, $\beta = \gamma = \frac{1}{2}$, $\delta = m+1$.

Its two solutions for $|z|<1$ are $y_1(z)$ and $y_2(z)$ where by VII (6)$_a$

18)$_a$ $y_1(z) \equiv f_1(z,\nu) \equiv F(a, a\nu - \frac{1}{4}(m+\frac{1}{2}); m+\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, m+1; z) =$

$= (1-k^2 z)^{-m}(1-z)^{\frac{1}{2}} F(a, a(\nu-\frac{1}{4})+\frac{1}{4}(m-\frac{1}{2}); 1-m, 1, \frac{1}{2}, 1-m; z)$ by VII (16)$_a$

$= (1-z)^{-\frac{1}{2}} F(a', a'[\nu-\frac{1}{4}(m-\frac{1}{2})]-\frac{1}{4}(m+\frac{1}{2}); 1, \frac{1}{2}, \frac{1}{2}, m+1; \frac{z}{z-1})$ by VII (14)

and (6)$_b$ where $a' \equiv 1/k'^2$.

And

18)$_b$ $y_2(z) \equiv z^{1/2} f_2(z,\nu) \equiv$

$= z^{\frac{1}{2}} F(a, a(\nu-\frac{1}{4})-\frac{1}{4}(3m+\frac{5}{2}); m+1, 1, \frac{3}{2}, m+1; z) =$

$= z^{\frac{1}{2}}(1-z)^{-1} F(a', a'[\nu-\frac{1}{4}(3m-\frac{5}{2})]-\frac{1}{4}(3m+\frac{5}{2}); 1, 3/2, 3/2, m+1; \frac{z}{z-1})$ by VII (14)

The relation VII (6)$_c$ takes the form

18)$_c$ $2z(1-z)^{1/2}[f_1(z,\nu) f_2'(z,\nu) - f_1'(z,\nu) f_2(z,\nu)] + (1-z)^{\frac{1}{2}} f_1(z,\nu) f_2(z,\nu) = \frac{1}{(1-k^2 z)^{m+1}}$

Since $\gamma+\delta-\alpha-\beta = 1/2$, both the Heun's functions $f_1(1,\nu)$ and $f_2(1,\nu)$ converge and therefore they are integral functions of ν. The functions $f_1(z,\nu)$ and $f_2(z,\nu)$, when $1-z$ is small, are of the form $A(z) + (1-z)^{\frac{1}{2}} B(z)$ where

$A(1)$ and $B(1)$ converge, so that $f_1'(z,\nu)$ and $f_2'(z,\nu)$ become infinite like $(1-z)^{-\frac{1}{2}}$ when $z \to 1$, i.e. in such a manner that the limits $(1-z)^{\frac{1}{2}} f_1'(z,\nu)$ and $(1-z)^{\frac{1}{2}} f_2'(z,\nu)$ $(z \to 1)$ exist. To express these limits in terms of f_1 and f_2 we make use of the solutions y_3 and y_4 in VII $(17)_a$ and $(17)_b$.

For the particular set of parameters listed here (in eq(17)) it is found that the functions F_3 and F_4 are the same functions as f_1 and f_2 respectively but with different arguments, so that $(17)_a$ and $(17)_b$ of section VII become

$$19)_a \quad y_3(z) = \left(\frac{1-K^2 z}{K'^2}\right)^{-m-\frac{1}{2}} f_1\left(\frac{1-z}{1-K^2 z}, \nu\right)$$

and

$$19)_b \quad y_4(z) = (1-z)^{\frac{1}{2}} \left(\frac{1-K^2 z}{K'^2}\right)^{-m-1} f_2\left(\frac{1-z}{1-K^2 z}, \nu\right)$$

Hence the equations $(19)_a$ and $(19)_b$ of VII become

$$20)_a \quad y_1(z) = f_1(z,\nu) =$$

$$= \left(\frac{1-K^2 z}{K'^2}\right)^{-m+\frac{1}{2}} \left\{ f_1(1,\nu) f_1\left(\frac{1-z}{1-K^2 z},\nu\right) + \frac{\left[I - K'^{2m+1} f_1^2(1,\nu)\right]}{K'^{2m+1}} \left(\frac{1-z}{1-K^2 z}\right)^{\frac{1}{2}} \frac{f_2\left(\frac{1-z}{1-K^2 z},\nu\right)}{f_1(1,\nu)} \right\}$$

and

$$20)_b \quad y_2(z) = z^{\frac{1}{2}} f_2(z,\nu) =$$

$$= \left(\frac{1-K^2 z}{K'^2}\right)^{-m-\frac{1}{2}} \left\{ f_2(1,\nu) f_1\left(\frac{1-z}{1-K^2 z},\nu\right) - f_1(1,\nu) \cdot \left(\frac{1-z}{1-K^2 z}\right)^{\frac{1}{2}} f_2\left(\frac{1-z}{1-K^2 z},\nu\right) \right\}$$

Some formulas will be required for the special case in which the argument, z, on the left of these equations is equal to that, $\frac{1-z}{1-K^2z}$, on the right, their common value being $\frac{1}{1+K'}$. Each of the equations $(20)_a$ and $(20)_b$ then reduces to

$$21)_a \quad \frac{(1+K')^{\frac{1}{2}} f_1\left(\frac{1}{1+K'}, \nu\right)}{f_2\left(\frac{1}{1+K'}, \nu\right)} = \frac{1 + K'^{m+\frac{1}{2}} f_1(1,\nu)}{K'^{m+\frac{1}{2}} f_2(1,\nu)}$$

If we differentiate $(20)_a$ or $(20)_b$ with respect to z and then place $z = \frac{1}{1+K'}$, we obtain by the use of the above relation

$$21)_b \quad \left[1 + K'^{m+\frac{1}{2}} f_1(1,\nu)\right] \frac{f_1'\left(\frac{1}{1+K'}, \nu\right)}{f_1\left(\frac{1}{1+K'}, \nu\right)} + \left[1 - K'^{m+\frac{1}{2}} f_1(1,\nu)\right] \frac{f_2'\left(\frac{1}{1+K'}, \nu\right)}{f_2\left(\frac{1}{1+K'}, \nu\right)} =$$
$$= \left(m+\tfrac{1}{2}\right)\frac{K^2}{K'} - \left(\frac{1+K'}{2}\right)\left[1 - K'^{m+\frac{1}{2}} f_1(1,\nu)\right].$$

Placing $z = 1/1+K'$ in eq $(18)_a$ gives another relation between the same functions and their derivatives. Between it and $(21)_b$ we find

$$22)_a \quad f_1'\left(\tfrac{1}{1+K'}, \nu\right) - \frac{(2m+1)K^2}{4K'} f_1\left(\tfrac{1}{1+K'}, \nu\right) = \frac{-(1+K')^{3/2}\left[1 - K'^{m+\frac{1}{2}} f_1(1,\nu)\right]}{4K'^{m+3/2} f_2\left(\frac{1}{1+K'}, \nu\right)}$$

and

$$22)_b \quad f_2'\left(\frac{1}{1+K'}, \nu\right) - \left[\frac{(2m+1)K^2}{4K'} - \frac{1+K'}{2}\right] f_2\left(\frac{1}{1+K'}, \nu\right) =$$

$$= \frac{(1+K')^2}{4} \frac{f_2(1,\nu)}{f_2\left(\frac{1}{1+K'}, \nu\right)}.$$

$$= \frac{(1+K')^{3/2}\left[1 + K'^{m+\frac{1}{2}} f_1(1,\nu)\right]}{4\, K'^{m+\frac{1}{2}} f_1\left(\frac{1}{1+K'}, \nu\right)}$$

An expression for the limits required, may now be found by differentiating $(20)_a$ with respect to z, multiplying by $(1-z)^{\frac{1}{2}}$ and then letting $z \to 1$ We find by use of $(21)_a$

$$23)_a \quad \lim_{z \to 1}\left[(1-z)^{\frac{1}{2}} f_1'(z,\nu)\right] = -\frac{\left[1 - K'^{2m+1} f_1^2(1,\nu)\right]}{2K'^{2m+2} f_2(1,\nu)} = \frac{-(1+K')^{\frac{1}{2}} f_1\left(\frac{1}{1+K'}, \nu\right) \cdot \left[1 - K'^{m+\frac{1}{2}} f_1(1,\nu)\right]}{2K'^{2m+2} f_2(1,\nu)}$$

$$= \frac{2}{1+K'} f_1\left(\frac{1}{1+K'}, \nu\right)\left[f_1'\left(\frac{1}{1+K'}, \nu\right) - \frac{(2m+1)K^2}{4K'} f_1\left(\frac{1}{1+K'}, \nu\right)\right] \text{ by } (22)_a$$

Then by $(18)_a$

$$23)_b \quad \lim_{z \to 1}\left[(1-z)^{\frac{1}{2}} f_2'(z,\nu)\right] = \frac{1}{2K'} f_1(1,\nu)$$

Taking $\omega = \alpha$, $z = \operatorname{sn}^2\frac{\alpha}{2}$, $u(\alpha) = \operatorname{dn}^{m+\frac{1}{2}}\frac{\alpha}{2}\, y(z)$ we get with $y_1(z)$ and $y_2(z)$, two solutions of $(11)_a$, one is the even function of α

$$24)_a \quad u = C(\alpha,\nu) \equiv \operatorname{dn}^{m+\frac{1}{2}}\frac{\alpha}{2}\, f_1\left(\operatorname{sn}^2\frac{\alpha}{2}, \nu\right),$$ the other is the odd function of α

$$24)_b \quad u = S(\alpha,\nu) \equiv 2\sqrt{\nu}\, \operatorname{sn}\frac{\alpha}{2}\, \operatorname{dn}^{m+\frac{1}{2}}\frac{\alpha}{2} \cdot f_2\left(\operatorname{sn}^2\frac{\alpha}{2}, \nu\right)$$

Comparison of fig 1 with fig 3 shows that to obtain

reduced potential of the form $V^{m} = u(\alpha)\, v(\beta)$ which have no double distributions at the dotted part of the cut $\overline{OA_2}$, $\overline{O'A_2'}$ corresponding to $\alpha = \mp 2K$, the function $u(\alpha)$ must satisfy

25)$_a$ $u(-2K) = u(2K)$

The remaining condition

25)$_b$ $u'(-2K) = u'(2K)$ is necessary to insure that there is no simple distribution there. The solutions and their derivatives must be continuous for $-2K < \alpha < 2K$ and since (11)$_a$ is of second order, the solutions will be periodic functions of α with period $4K$ or $2K$.

The even functions of the form $C(\alpha, \nu)$ satisfy (25)$_a$ and since their derivatives are odd functions of α, the condition (25)$_b$ require that the corresponding characteristic values of ν say ν^c shall be solutions of $C'(2K, \nu^c) = 0$, i.e. $\lim_{z \to 1} \left[(1-z)^{\frac{1}{2}} f_1'(z, \nu^c)\right] = 0$, or, by the last form of (23)a, the necessary and sufficient condition becomes

$$26) \quad f_1\left(\tfrac{1}{1+K'}, \nu^c\right)\left\{ f_1'\left(\tfrac{1}{1+K'}, \nu^c\right) - \frac{(2m+1)K^2}{4K'} f_1\left(\tfrac{1}{1+K'}, \nu^c\right)\right\} = 0$$

The derivative of odd functions of the form $S(\alpha, \nu)$, being even will satisfy (25)$_b$ so that (25)$_a$ requires $S(2K, \nu^s) = 0$ or by (22)$_b$

$$27) \quad \left(\tfrac{1+K'}{2}\right)^2 f_2(1, \nu^s) = f_2\left(\tfrac{1}{1+K'}, \nu^s\right)\left\{ f_2'\left(\tfrac{1}{1+K'}, \nu^s\right) - \left[\frac{(2m+1)K^2}{4K'} - \frac{1+K'}{2}\right] f_2\left(\tfrac{1}{1+K'}, \nu^s\right)\right\} = 0$$

From the theory of integral equations it is known that there are an infinite number of real roots of these equations, and further that if $0 < \kappa < 1$ they are all of rank 1 so that one and only one eigen-function belongs to each eigen-value of ν. Also no eigen-values ν^c could be equal to one of type ν^s.

The set of eigen-values ν_m^c may be divided into the class ν_{2n}^c $(n = 0, 1, 2 \cdots \infty)$ belonging to the function $C_{2n}^m(\alpha, \kappa) \equiv C(\alpha, \nu_{2n}^c)$ of period $2K$, and the class ν_{2n-1}^c $(n = 1, 2, 3, \cdots \infty)$ belonging to the function $C_{2n-1}^m(\alpha, \kappa) \equiv C(\alpha, \nu_{2n-1}^c)$ of period $4K$.

Similarly the set ν_m^s consists of the set ν_{2n}^s $(n = 1, 2, 3, \cdots \infty)$ with functions $S_{2n}^m(\alpha, \kappa) \equiv S(\alpha, \nu_{2n}^s)$ of period $2K$, and the set ν_{2n-1}^s $(n = 1, 2, 3 \cdots)$ belonging to function $S_{2n-1}^m(\alpha, \kappa) \equiv S(\alpha, \nu_{2n-1}^s)$ with period $4K$. This notation suggests the periodicity and evenness or oddness by analogy of $C_n^m(\alpha, \kappa)$ with $\cos n\alpha$ and of $S_n^m(\alpha, \kappa)$ with $\sin n\alpha$, which become identities when $\kappa \to 0$, irrespective of the integer m. (Eq (32) below).

We may now identify the type of characteristics determined by the vanishing of each factor in (26) and similarly for (27).

The even function $C_{2n}^m(\alpha, \kappa)$ satisfies $(25)_a$; its derivative $C_{2n}'^m(\alpha, \kappa)$ being odd and also of period $2K$, must satisfy $C_{2n}'^m(K, \kappa) = 0$,

that is, $C'(\mathbb{K}, \nu^c_{2n})=0$. Since $cn\frac{\mathbb{K}}{2}=\sqrt{\frac{K'}{1+K'}}$, $sn\frac{\mathbb{K}}{2}=\frac{1}{\sqrt{1+K'}}$ and $dn\frac{\mathbb{K}}{2}=\sqrt{K'}$, this becomes by $(24)_a$ the vanishing of the parenthesis in (26). The other factor determines characteristics of type ν^c_{2n-1}. Similarly $\underset{2n}{S}^m(\alpha,\kappa)$ being odd and of period $2\mathbb{K}$ gives $\underset{2n}{S}^m(\mathbb{K},\kappa)=0$, that is, $f_2(\frac{1}{1+K'}, \nu^s_{2n})=0$. The vanishing of the parenthesis in (27) determines λ^s_{2n-1}. Hence (26) and (27) break up into the following four characteristic equations

$$28)_a \quad f_1'\left(\frac{1}{1+K'}, \nu^c_{2n}\right)-\frac{(2m+1)K^2}{4K'}f_1\left(\frac{1}{1+K'}, \nu^c_{2n}\right)=0 \qquad n=0,1,2,3,\dots\infty$$

$$28)_b \quad f_1\left(\frac{1}{1+K'}, \nu^c_{2n-1}\right)=0 \qquad n=1,2,3,\dots\infty$$

$$28)_c \quad f_2\left(\frac{1}{1+K'}, \nu^s_{2n}\right)=0 \qquad n=1,2,3\dots\infty$$

$$28)_d \quad f_2'\left(\frac{1}{1+K'}, \nu^s_{2n-1}\right)-\left[\frac{(2m+1)K^2}{4K'}-\frac{1+K'}{2}\right]f_2\left(\frac{1}{1+K'}, \nu^s_{2n-1}\right)=0 \qquad n=1,2,3\dots\infty$$

These equations are unambiguous; every root of one of them is a characteristic value ν of the type indicated by upper and lower indices of the ν and there are no other characteristics of that type.

On the other hand the equation $K'^{\,m+\frac{1}{2}} f_1(1,\nu)=1$ has for its roots all of the set ν^c_{2n} and of the set ν^s_{2n} but no other roots. This may be put in evidence by

writing

29)$_a$ $\kappa'^{m+\frac{1}{2}} f_1(1, \nu_{2n}) = 1$. Similarly the equation,

29)$_b$ $\kappa'^{m+\frac{1}{2}} f_1(1, \nu_{2n-1}) = -1$, indicates that all its roots include the set ν^c_{2n-1} and ν^s_{2n-1}. To prove this; (29)$_a$ follows from writing (28)$_a$, by use of 22)$_a$, in the form

$$\frac{1 - \kappa'^{m+\frac{1}{2}} f_1(1, \nu^c_{2n})}{f_2(\frac{1}{1+\kappa'}, \nu^c_{2n})} = 0.$$

The denominator does not vanish since ν^c_{2n} and ν^s_{2n} are distinct. It only vanishes when $\nu = \nu^s_{2n}$ as shown by 28)$_c$. By the same argument (28)$_c$ with (28)$_a$ show that ν^s_{2n} also satisfies (29)$_a$. That ν^c_{2n-1} satisfies (29)$_b$ is shown by (28)$_b$ and (29)$_b$. Similarly (28)$_d$ with (29)$_b$ shows that equation (29)$_b$ is satisfied by ν^s_{2n-1}.

From the characteristic equations and the definitions and preceding equations we may now prove the following relations

30)$_a$ $C^m_n(2K-\alpha, \kappa) = (-1)^n C^m_n(\alpha, \kappa)$

30)$_b$ $S^m_n(2K-\alpha, \kappa) = (-1)^{n+1} S^m_n(\alpha, \kappa)$ which are analogous to $\cos n(\pi-\alpha) = (-1)^n \cos n\alpha$ and $\sin n(\pi-\alpha) = (-1)^{n+1} \sin n\alpha$

To prove them we note that in the case where n is even they are consequences of the periodicity $(2K)$ together with the evenness or oddness of the functions, but when n is odd these relations imply a further symmetry not thus

derivable. To prove (30) we use the equation $(20)_a$ making use of the fact that ν^c_{2n} satisfies $(29)_a$ and ν^c_{2n-1} satisfies $(29)_b$ but neither satisfies (27). This gives

$$C^m_n(\alpha,\kappa) = (-1)^n \left(\frac{\kappa'}{dn\frac{\alpha}{2}}\right)^{m+\frac{1}{2}} f_1\left(\frac{cn^2\alpha/_2}{dn^2\alpha/_2}, \nu^c_n\right) = (-1)^n C^m_n(2K-\alpha,\kappa)$$

since

$$dn(K-\tfrac{\alpha}{2}) = \frac{\kappa'}{dn\frac{\alpha}{2}} \text{ and } sn(K-\tfrac{\alpha}{2}) = \frac{cn\frac{\alpha}{2}}{dn\frac{\alpha}{2}}.$$

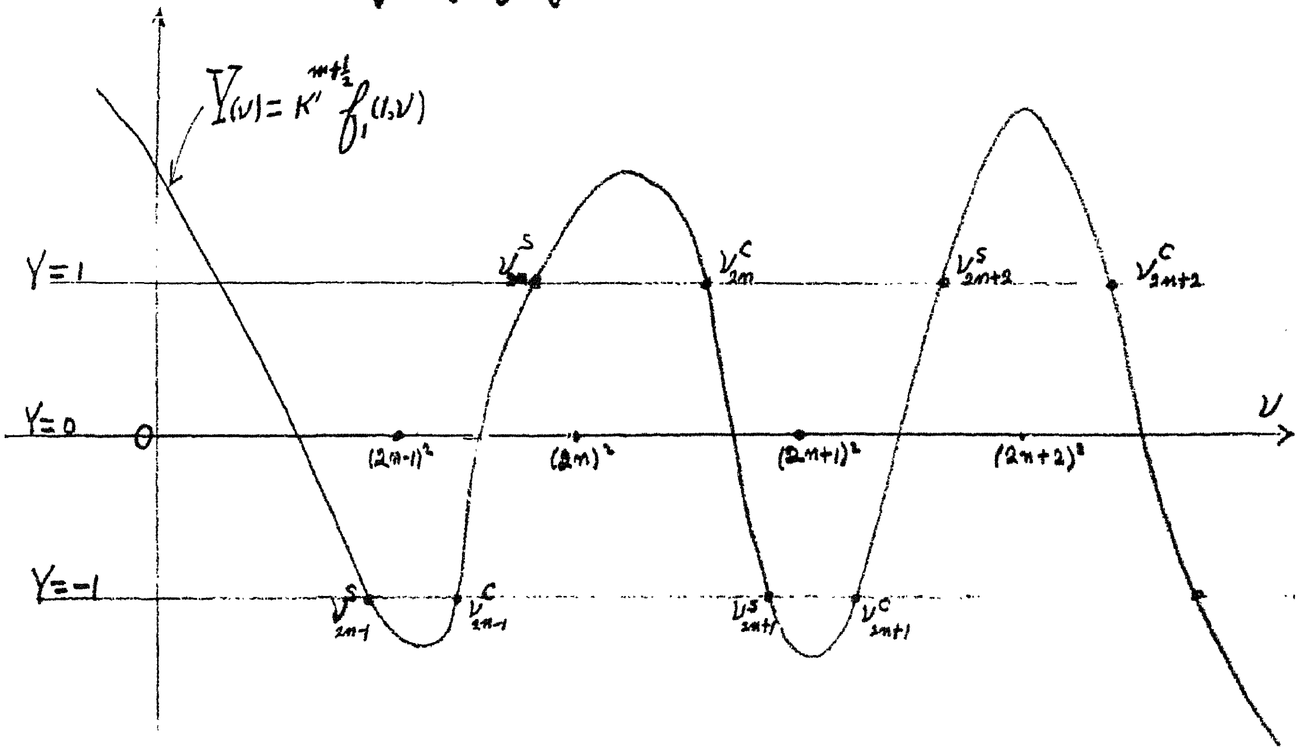
Similarly $(20)_b$ gives $(30)_b$ since ν^s_n satisfies (27).

By reason of (30) it would be sufficient to make a table (for each fixed value of κ and of the integer m) of the functions $C^m_n(\alpha,\kappa)$ and $S^m_n(\alpha,\kappa)$ for the range $0 \leq \alpha \leq K$.

The following table follows from the definitions and the relations (30).

α	C_{2n}	C'_{2n}	S_{2n}	S'_{2n}	C_{2n-1}	C'_{2n-1}	S_{2n-1}	S'_{2n-1}
0	1	0	0	$\sqrt{\nu^s_{2n}}$	1	0	0	$\sqrt{\nu^s_{2n-1}}$
K	$\neq 0$	0	0	$\neq 0$	0	$\neq 0$	$\neq 0$	0
$2K$	1	0	0	$\sqrt{\nu^s_{2n}}$	-1	0	0	$-\sqrt{\nu^s_{2n-1}}$

If for a given m and κ the integral function of ν $Y(\nu) \equiv \kappa'^{m+\frac{1}{2}} f_1(1,\nu)$ be plotted against ν the eigen-values are the abscissae where this curve crosses the horizontal lines $Y = \pm 1$ (eq (29)$_a$ (29)$_b$). The various classes of characteristics are indicated on the accompanying figure



In the limiting case where $a_2 \to a_1$, $\kappa' \to 1$, $\kappa = 0$, $K \to \frac{\pi}{2}$ and $K' \to \infty$ and eq(1) becomes $z = -a_1 \cot \frac{u}{2}$ as in Toroidal coordinates. Since $a = 1/\kappa^2 \to \infty$, the definition (18)$_a$ of f_1 becomes by VII (9)$_b$ (since $\operatorname{cn} \alpha/2 \to \cos\frac{\alpha}{2}$, $\operatorname{sn}\frac{\alpha}{2} \to \sin\frac{\alpha}{2}$ and $\operatorname{dn}\frac{\alpha}{2} \to 1$)

31)$_a$ $f_1(\operatorname{sn}^2\frac{\alpha}{2}, \nu) \to F(\sqrt{\nu}, -\sqrt{\nu}, \frac{1}{2}; \sin^2\frac{\alpha}{2}) = \cos\alpha\sqrt{\nu}$ by I (6)$_e$

31)$_b$ $f_2(\operatorname{sn}^2\frac{\alpha}{2}, \nu) \to F(\frac{1}{2}+\sqrt{\nu}, \frac{1}{2}-\sqrt{\nu}, \frac{3}{2}; \sin^2\frac{\alpha}{2}) = \dfrac{\sin\alpha\sqrt{\nu}}{2\sqrt{\nu}\sin\frac{\alpha}{2}}$ by I (6)$_b$

The equation of the curve in the preceding figure becomes $Y(\nu) = \cos\alpha\sqrt{\nu}$ which is tangent to the lines $Y = \pm 1$; the characteristics become $\nu^c_{2n} = \nu^s_{2n} = (2n)^2$ when $Y(\nu) = +1$ and $\nu^c_{2n-1} = \nu^s_{2n-1} = (2n-1)^2$. The rank of the characteristics is 2 except for $n = 0$, and for $\nu = n > 0$ the two eigen-functions are

32) $\quad C^m_n(\alpha, 0) = \cos n\alpha$ and $S^m_n(\alpha, 0) = \sin n\alpha$.

which were found for toroidal coordinates.

From the differential equation $(11)_a$ and the boundary conditions enumerated above, one obtains an integral as in IX (39) which shows that for the range $0 < \alpha < K$ the functions $C^m_{2n}(\alpha, K)$ $(n = 0, 1, 2, \dots \infty)$ form a complete set of orthogonal functions. Another set is $S^m_{2n}(\alpha, K)$ $(n = 1, 2, \dots \infty)$ while set $(C^m_{2n}) + S^m_{2n}$ is complete for the range $-K < \alpha < K$. For the large positive range $0 < \alpha < 2K$ one complete set is $(C^m_n) = (C^m_{2n}) + (C^m_{2n-1})$. Another is $(S^m_n) = (S^m_{2n}) + (S^m_{2m-1})$

Functions of all four classes, i.e. $(C^m_n) + (S^m_n)$ are required to make a complete set for $-2K < \alpha < 2K$. When they are normalized for this range, they may be denoted by $\mathfrak{c}^m_n(\alpha, K) = A^m_n C^m_n(\alpha, K)$ and $\mathfrak{s}^m_n(\alpha, K) = B^m_n S^m_n(\alpha, K)$ where the constants A and B are so chosen as to give the normal conditions

$$33)\quad \begin{cases} \displaystyle\int_{-2K}^{2K} [C_n^m(\alpha, \kappa)]^2 d\alpha = \int_{-2K}^{2K} [S_n^m(\alpha,\kappa)]^2 d\alpha = 1 \,, \\ \displaystyle\int_{-2K}^{2K} C_{n_1}^m(\alpha,\kappa)\, C_{n_2}^m(\alpha,\kappa)\, d\alpha = \int_{-2K}^{2K} S_{n_1}^m(\alpha,\kappa)\, S_{n_2}^m(\alpha,\kappa)\, d\alpha = 0 \text{ if } n_1 \neq n_2 \,, \\ \text{and } \displaystyle\int_{-2K}^{2K} C_n^m(\alpha,\kappa)\, S_{n'}^m(\alpha,\kappa)\, d\alpha = 0 \text{ for all } n \text{ and } n' . \end{cases}$$

To complete the sketch of the potential problem which assigns values on the circular annulus (or on its inversion, fig 2) it remains to examine the solutions $v(\beta)$ of the equation (11)$_e$ and to assign to them their inconspicuous part. It may be noted however that under happier circumstances they too could blossom forth as normal functions, for example when the potential is assigned on all parts of the plane $x=0$ except on the annulus which is then regarded as an annular aperture between a circular disc and its "guard plane, as in the absolute electrometer.

From the two solutions of (1 b), $y_1(z)$ and $y_2(z)$ given in 18)$_a$ and (18)$_b$ we get two solutions of (11)$_e$ by taking

$$\omega = i(\beta - 2K'),\ dn\frac{\omega}{2} = \frac{1}{sn\,\beta/2},\ sn\frac{\omega}{2} = \frac{cn\,\beta/2}{i\kappa\, sn\,\beta/2} \text{ and } cn\frac{\omega}{2} = \frac{dn\,\beta/2}{\kappa\, sn\,\beta/2}$$

and $z = sn^2\frac{\omega}{2} = -\frac{cn^2\beta/2}{\kappa^2 sn^2\beta/2}$ where the modulus for all the functions of β is κ'.

The first is $\mathcal{V}^{cm}(\beta,\nu) \equiv sn\beta_{/2}^{-m-\frac{1}{2}} f_1\left(-\frac{cn^2\beta_{/2}}{K^2 sn^2\beta_{/2}}, \nu\right)$ or by the last form of (18)$_a$

34)$_a$ $$\mathcal{V}^{cm}(\beta,\nu) \equiv \frac{K\, sn\beta_{/2}}{dn\beta_{/2}}^{\frac{1}{2}-m} F\left(a', -a'\left[\nu-\tfrac{1}{4}(m-\tfrac{1}{2})\right]-\tfrac{1}{4}(m+\tfrac{1}{2}); 1, \tfrac{1}{2}, \tfrac{1}{2}, m+1; \frac{cn^2\beta_{/2}}{dn^2\beta_{/2}}\right)$$

The second is $\mathcal{V}^{sm}(\beta,\nu) \equiv \frac{2}{K} cn\beta_{/2}\, sn\beta_{/2}^{-m-3/2} f_2\left(-\frac{cn^2\beta_{/2}}{K^2 sn^2\beta_{/2}}, \nu\right)$ or by the last form of (18)$_b$

34)$_b$ $$\mathcal{V}^{sm}(\beta,\nu) \equiv \frac{2K\, cn\beta_{/2}}{dn^2\beta_{/2}} \cdot sn\beta_{/2}^{\frac{1}{2}-m} F\left(a', -a'\left[\nu-\tfrac{1}{4}(3m+\tfrac{5}{2})\right]-\tfrac{1}{4}(3m+\tfrac{5}{2}); 1, \tfrac{3}{2}, \tfrac{3}{2}, m+1; \frac{cn^2\beta_{/2}}{dn^2\beta_{/2}}\right)$$

Hence

35)$_a$ $$\mathcal{V}^{cm}(2K',\nu) = 1 \quad \text{and} \quad \mathcal{V}'^{cm}(2K',\nu) = 0.$$

35)$_b$ $$\mathcal{V}^{sm}(2K',\nu) = 0 \quad \text{and} \quad \mathcal{V}'^{sm}(2K',\nu) = -1$$

We require a third solution of (16) which is y_5 of section VII eq(21)$_b$ with argument $\frac{a}{a-z} = sn^2\beta_{/2}$. It becomes

36) $$\begin{cases} \mathcal{V}^{om}(\beta,\nu) = sn\beta_{/2}^{m+\frac{1}{2}} F_0(sn^2\beta_{/2},\nu) \quad \text{where} \\ F_0(sn^2\beta_{/2},\nu) \equiv F\left(a', -a'\left[\nu+\tfrac{1}{4}(m+\tfrac{1}{2})\right]-\tfrac{1}{4}(m+\tfrac{1}{2}); m+\tfrac{1}{2}, \tfrac{1}{2}, m+1, \tfrac{1}{2}; sn^2\beta_{/2}\right) \end{cases}$$

The other solution for this range involves logarithms since γ is an integer, $m+1$. Hence (36) is the only solution of (11)$_b$ satisfying the condition on the x-axis ($\beta = 0$), that reduced potentials of the form $\mathcal{V} = \mathcal{U}(\alpha)\mathcal{V}(\beta)$ must vanish like $\rho^{m+\frac{1}{2}}$.

The function $F'_0(1,\nu)$ converges since $\gamma+\delta-\alpha-\beta=\frac{1}{2}$, but $F'_0(z,\nu)$ becomes infinite in such a manner that the $\lim_{z\to 1}$ of $(1-z)^{\frac{1}{2}} F'_0(z,\nu)$ exists. The limits

37) $\gamma^{sm}(\nu) \equiv \mathfrak{V}'^{om}(2K',\nu) =$ and $\gamma^{cm}(\nu) \equiv \mathfrak{V}'^{cm}(2K',\nu) = \lim_{z\to 1}(1-z)^{\frac{1}{2}} F'_0(z,\nu)$

both exist and are different from zero.

The three solutions of $(11)_b$ are connected by the linear relation

38) $\mathfrak{V}^{om}(\beta,\nu) = \gamma^{sm}(\nu)\,\mathfrak{V}^{cm}(\beta,\nu) - \gamma^{cm}(\nu)\,\mathfrak{V}^{sm}(\beta,\nu)$

Also

$39)_a$ $\gamma^{cm}(\nu) = \mathfrak{V}^{cm}(\beta,\nu)\,\mathfrak{V}'^{om}(\beta,\nu) - \mathfrak{V}'^{cm}(\beta,\nu)\,\mathfrak{V}^{om}(\beta,\nu)$

and

$39)_b$ $\gamma^{sm}(\nu) = \mathfrak{V}^{sm}(\beta,\nu)\,\mathfrak{V}'^{om}(\beta,\nu) - \mathfrak{V}'^{sm}(\beta,\nu)\,\mathfrak{V}^{om}(\beta,\nu)$

The functions $\mathfrak{V}^c(\beta)$ and $\mathfrak{V}^s(\beta)$ are for internal use and $\mathfrak{V}^o(\beta)$ for external use only.

The (reduced) potential $U^m(\alpha,\beta)$ of a simple distribution with (reduced) density $\bar{\sigma}(\alpha)$ on the closed curve in the $x\rho$ half plane of fig 1 (or on its inversion in fig 2) whose equation is $\beta=\beta_1$, is given at all points (α,β) by the potential integral

$40)_a$ $U^m(\alpha,\beta) = 2\displaystyle\int_{-2K}^{2K} \frac{\bar{\sigma}(\alpha_1)}{h(\alpha_1,\beta_1)}\, Q_{m-1/2}\big(g(\alpha,\beta;\alpha_1,\beta_1)\big)\, d\alpha_1$ where h is

given by (3) and g by

$$40)_b \quad g(\alpha,\beta;\alpha_1,\beta_1) = 1 + \frac{(x-x_1)^2+(\rho-\rho_1)^2}{2\rho\rho_1} =$$

$$= \frac{dn^2\alpha/_2\, sn^2\beta/_2 + dn^2\alpha_1/_2\, sn^2\beta_1/_2 + \left[sn\alpha/_2\, dn\beta/_2\, cn\frac{\alpha_1}{2}\, cn\frac{\beta_1}{2} - sn\frac{\alpha_1}{2}\, dn\frac{\beta_1}{2}\, cn\frac{\alpha}{2}\, cn\frac{\beta}{2}\right]^2}{2\, dn\frac{\alpha}{2}\, sn\frac{\beta}{2}\, dn\frac{\alpha_1}{2}\, sn\frac{\beta_1}{2}}$$

the modulus associated with α and α_1 being κ, that with β and β_1 being κ'.

If the internal potential is $U^{im}_{(\alpha,\beta)}$ for $\beta_1 \leq \beta \leq 2K'$ and the external is $U^{om}_{(\alpha,\beta)}$ for $0 \leq \beta \leq \beta_1$, the density is given by

$$40)_c \quad \frac{4\pi\bar{\sigma}(\alpha)}{h(\alpha,\beta_1)} = -\left(D_\beta U^{im}\right)_{\beta\to\beta_1+0} + \left(D_\beta U^{om}\right)_{\beta\to\beta_1-0}$$

The potential is continuous at $\beta=\beta_1$.

To construct external harmonics, it is evident that the condition for no sources on the x axis permits only of the solution $\mathcal{V}^{om}(\beta,\nu)$ where ν must be the characteristic or eigen-value of the periodic function which is associated with $\mathcal{V}^{om}$ as a factor. Hence the external harmonics must be of the form

$$U^{om} = A\, C^m_n(\alpha,\kappa)\, \mathcal{V}^{om}(\beta,\nu^c_n) + B\, S^m_n(\alpha,\kappa)\, \mathcal{V}^{om}(\beta,\nu^s_n)$$

The boundary conditions determining $C^m_n(\alpha)$ and $S^m_n(\alpha)$ insure that there is neither simple nor double distribution at the part of the cut shown dotted in fig 1 and fig 2 $\overline{OB_2A_2}$ $(\alpha=-2K)$ and $\overline{O'B_2'A_2'}$ $(\alpha=2K)$. This character extends also

to the internal harmonics which must in addition be so chosen that the heavy part of the cut $A_2A_1A_3'$ ($\beta = 2K'$), that is, the circular annulus or its inversion, shall not be the seat of charges, either simple or double distributions.

An internal potential of the form $C_n^m(\alpha,\kappa)\,\mathcal{V}^{sm}(\beta,\nu_n^c)$ would vanish on both sides of the annulus ($\beta = 2K'$), but would represent a simple distribution there, and the form $S^m(\alpha,\kappa)\,\mathcal{V}^{cm}(\beta,\nu_n^s)$ a double distribution. Hence the normal solutions are of the form

41)$_a$ $$U_{cn}^{im}(\alpha,\beta) = c_n^m(\alpha)\,\frac{\mathcal{V}^{cm}(\beta,\nu_n^c)}{\mathcal{V}^{cm}(\beta_1,\nu_n^c)} \qquad \text{where} \quad \beta_1 \leq \beta \leq 2K'.$$

41)$_b$ $$U_{cn}^{om}(\alpha,\beta) = c_n^m(\alpha)\,\frac{\mathcal{V}^{om}(\beta,\nu_n^c)}{\mathcal{V}^{om}(\beta_1,\nu_n^c)} \qquad \text{where} \quad 0 \leq \beta \leq \beta_1 .$$

The density is

41)$_c$ $$\frac{4\pi\,\bar{\sigma}_c(\alpha)}{h(\alpha,\beta_1)} = \frac{\gamma^{cm}(\nu_n^c)}{\mathcal{V}^{om}(\beta_1\nu_n^c)\,\mathcal{V}^{cm}(\beta_1,\nu_n^c)}\cdot c_n^m(\kappa) \equiv 4\pi\lambda_n^{cm}(\beta_1)\cdot c_n^m(\alpha).$$

And the odd functions of α are

42)$_a$ $$U_{sn}^{im}(\alpha,\beta) = s_n^m(\alpha)\,\frac{\mathcal{V}^{sm}(\beta,\nu_n^s)}{\mathcal{V}^{sm}(\beta_1,\nu_n^s)} \qquad \text{where} \quad \beta_1 \leq \beta \leq 2K'$$

42)$_b$ $$U_{sn}^{om}(\alpha,\beta) = s_n^m(\alpha)\,\frac{\mathcal{V}^{om}(\beta,\nu_n^s)}{\mathcal{V}^{om}(\beta_1,\nu_n^s)} \qquad \text{where} \quad 0 \leq \beta \leq \beta_1$$

The density is

42)$_c$ $$\frac{4\pi\,\bar{\sigma}_s(\alpha)}{h(\alpha,\beta_1)} = \frac{\gamma^{sm}(\nu_n^s)}{\mathcal{V}^{om}(\beta_1,\nu_n^s)\,\mathcal{V}^{sm}(\beta_1,\nu_n^s)}\cdot s_n^m(\alpha) \equiv 4\pi\lambda_n^{sm}(\beta_1)\,s_n^m(\alpha).$$

where c, s are the normalized forms of $C(\alpha)$, $S(\alpha)$ as in (33).

With these densities and potentials the integral $(40)_a$ becomes the following homogeneous linear integral equation with $Q_{m-\frac{1}{2}}(q)$ as nucleus which is satisfied by every normal function $c_n^m(\alpha)$, $s_n^m(\alpha)$ and hence by $C_n^m(\alpha,\kappa)$ and $S_n^m(\alpha,\kappa)$

$$43)_a \quad \int_{-2K}^{2K} C_n^m(\alpha_1,\kappa)\, Q_{m-1/2}\big(g(\alpha,\beta;\alpha_1,\beta_1)\big)\,d\alpha_1 =$$

$$= \frac{2\pi\, \overset{om}{V}(\beta_1,\nu_n^c)\, \overset{cm}{V}(\beta,\nu_n^c)}{\gamma^{cm}(\nu_n^c)}\, C_n^m(\alpha,\kappa) \qquad \text{if} \quad \beta_1 \leq \beta \leq 2K'$$

$$= \frac{2\pi\, \overset{cm}{V}(\beta_1,\nu_n^c)\, \overset{o,m}{V}(\beta,\nu_n^c)}{\gamma^{cm}(\nu_n^c)}\, C_n^m(\alpha,\kappa) \qquad \text{if} \quad 0 \leq \beta \leq \beta_1$$

$$43)_b \quad \int_{-2K}^{2K} S_n^m(\alpha_1,\kappa)\, Q_{m-1/2}\big(g(\alpha,\beta;\alpha_1,\beta_1)\big)\,d\alpha_1 =$$

$$= \frac{2\pi\, \overset{om}{V}(\beta_1,\nu_n^s)\, \overset{sm}{V}(\beta,\nu_n^s)}{\gamma^{sm}(\nu_n^s)}\, S_n^m(\alpha,\kappa) \qquad \text{if} \quad \beta_1 \leq \beta \leq 2K'$$

$$= \frac{2\pi\, \overset{sm}{V}(\beta_1,\nu_n^s)\, \overset{om}{V}(\beta,\nu_n^s)}{\gamma^{sm}(\nu_n^s)}\, S_n^m(\alpha,\kappa) \qquad \text{if} \quad 0 \leq \beta \leq \beta_1$$

If the points (α,β) and (α_1,β_1) are both on the annulus or its inverse, $\beta=\beta_1=2K'$, and eq $(43)_a$ becomes

$$43)_c \quad \int_0^{2K} C_n^m(\alpha_1,\kappa)\, Q_{m-1/2}\left(\frac{1}{2}\left(\frac{dn\frac{\alpha}{2}}{dn\frac{\alpha_1}{2}} + \frac{dn\frac{\alpha_1}{2}}{dn\frac{\alpha}{2}}\right)\right)d\alpha_1 = \pi\,\frac{\gamma^{sm}(\nu_n^c)}{\gamma^{cm}(\nu_n^c)}\, C_n^m(\alpha,\kappa)$$

Where $\gamma^{sm}(\nu)$ and $\gamma^{cm}(\nu)$ are defined by $(39)_a$ and $(39)_b$. This

equation is analogous to Whitaker's integral equation satisfied by the Lamé-Hermite polynomials: Modern Analysis, 4th edition, pp. 564-567.

The formal development theorem is

$$44)\quad f(\alpha) = \sum_{n=0}^{\infty} \int_{-2K}^{2K} f(\alpha') \left[\mathcal{C}_n^m(\alpha)\, \mathcal{C}_n^m(\alpha') + \mathcal{S}_n^m(\alpha)\, \mathcal{S}_n^m(\alpha') \right] d\alpha' \quad \text{for } -2K < \alpha < 2K$$

Developing $Q_{m-1/2}(g)$ as a function of α, considering $\alpha_1, \beta,$ and β_1 as constants, the coefficients of the series are given by the integral equations (43). This gives the following expansion of the symmetric nucleus $Q_{m-1/2}(g)$ as an addition theorem of the general form obtained in IX (44)

$$45)\quad Q_{m-1/2}(g(\alpha,\beta;\alpha_1,\beta_1)) =$$

$$= 2\pi \sum_{n=0}^{\infty} \left[\frac{\mathcal{C}_n^m(\alpha)\, \mathcal{V}^{om}(\beta, \nu_n^c)\, \mathcal{C}_n^m(\alpha_1)\, \mathcal{V}^{cm}(\beta_1, \nu_n^c)}{\gamma^{cm}(\nu_n^c)} + \frac{S_n^m(\alpha)\, \mathcal{V}^{om}(\beta, \nu_n^s)\, S_n^m(\alpha_1)\, \mathcal{V}^{sm}(\beta_1, \nu_n^s)}{\gamma^{sm}(\nu_n^s)} \right]$$

when $0 \le \beta \le \beta_1$, these being interchanged otherwise.

The special case of this, when the point (α_1, β_1) is on the annulus, $\beta_1 = 2K'$, is

$$45)'\quad Q_{m-1/2}\left(\frac{1 - sn^2\frac{\alpha}{2}\, dn^2\frac{\beta}{2} + \left(1 - cn^2\frac{\alpha}{2}\, cn^2\frac{\beta}{2}\right) dn^2\frac{\alpha_1}{2}}{2\, dn\frac{\alpha}{2}\, sn\frac{\beta}{2}\, dn\frac{\alpha_1}{2}} \right) = 2\pi \sum_{n=0}^{\infty} \frac{\mathcal{C}_n^m(\alpha)\, \mathcal{C}_n^m(\alpha_1)\, \mathcal{V}^{om}(\beta, \nu_n^c)}{\gamma^{cm}(\nu_n^c)}$$

which is valid everywhere.

If the point (α,β) is also on the annulus $\beta = 2K'$, this becomes

$$45)'' \quad Q_{m-1/2}\left(\tfrac{1}{2}\left(\frac{dn\frac{\alpha}{2}}{dn\frac{\alpha_1}{2}} + \frac{dn\frac{\alpha_1}{2}}{dn\frac{\alpha}{2}}\right)\right) = \sum_{n=0}^{\infty} \frac{\mathcal{C}_n^m(\alpha)\,\mathcal{C}_n^m(\alpha_1)}{\lambda_n^{c\,m}(2K')}$$

where $\lambda_n^{c\,m}(2K') = \dfrac{\gamma^{om}(\nu_n^c)}{2\pi\, \mathcal{V}^{om}(2K',\nu_n^c)} = \dfrac{\mathcal{V}'^{om}(2K',\nu_n^c)}{2\pi\,\mathcal{V}^{om}(2K',\nu_n^c)}$ by $(39)_c$ and $(35)_c$.

This is positive by IX (32).

The reduced potential $V(\alpha,\beta)$ which has given values $F(\alpha)$ on both sides of the annulus (or on its inversion) is given everywhere by

$$46) \quad \overset{m}{V}(\alpha,\beta) = \sum_{n=0}^{\infty} \int_{-2K}^{2K} F(\alpha_1)\left[\frac{\mathcal{V}^{om}(\beta,\nu_n^c)}{\mathcal{V}^{om}(2K',\nu_n^c)}\,\mathcal{C}_n^m(\alpha)\,\mathcal{C}_n^m(\alpha_1) + \frac{\mathcal{V}^{om}(\beta,\nu_n^s)}{\mathcal{V}^{om}(\beta_1,\nu_n^s)}\,S_n^m(\alpha)\,S_n^m(\alpha_1)\right]d\alpha_1$$

In the limiting case, $\kappa \to 1$, the annular coordinate system (α,β), degenerates into an inversion of oblate spheroidal coordinates. The development of an arbitrary function $f(\alpha)$, for the range $-K < \alpha < K$ in a series of normal functions $\mathcal{C}_{2n}^m(\alpha,\kappa)$, $\mathcal{S}_{2n}^m(\alpha,\kappa)$ is replaced by the integral representation of the form given in VIII $(31)_c$. The eigen-values $\nu_{2n}^c(\kappa)$ and $\nu_{2n}^s(\kappa)$ do not remain distinct but merge into a line. The

range $-\bar{K} < \alpha < \bar{K}$ becomes $-\infty < \alpha < \infty$, and by VII (9) the Heun's functions degenerate into hypergeometric functions, so that the definitions of $C(\alpha,\nu)$ and $S(\alpha,\nu)$ in $(24)_a$ and $(24)_b$ become (for $k \to 1$)

$$
(47)\quad \begin{cases} C(\alpha,\nu) \to \operatorname{sech}^{\mu}\frac{\alpha}{2}\, F\left(\frac{1}{4}+\frac{m}{2}+\frac{\mu}{2}, \frac{1}{4}-\frac{m}{2}+\frac{\mu}{2}, \frac{1}{2}; \tanh^2\frac{\alpha}{2}\right) \\[2ex] S(\alpha,\nu) \to 2\sqrt{\nu}\tanh\frac{\alpha}{2}\operatorname{sech}^{\mu}\frac{\alpha}{2}\, F\left(\frac{3}{4}+\frac{m}{2}+\frac{\mu}{2}, \frac{3}{4}-\frac{m}{2}+\frac{\mu}{2}, \frac{3}{2}; \tanh^2\frac{\alpha}{2}\right) \\ \text{where} \\ \mu = \sqrt{m^2-\frac{1}{4}-4\nu} \end{cases}
$$

These are proportional to the functions C^{μ}_{m} and S^{μ}_{m} (with β replaced by $\frac{\alpha}{2}$) defined in oblate spheroidal coordinates by equations $(26)_c$ and $(26)_d$. Considering the case $k=1$ as the case $a_1 \to \infty$ (a_2 fixed), the annulus becomes the infinite plane $x=0$ in which there is a circular aperture of radius a_2. This is also a limiting case of a hyperboloid of revolution so the potential with prescribed values on the annulus goes over into that having assigned values on a one-sheeted hyperboloid of oblate spheroidal coordinates.